

## Derivation of RK2 or Midpoint Method for Numerical Integration

Taylor series for time step of  $x(t)$

$$x(t+h) = x(t) + h \dot{x} + \frac{h^2}{2} \ddot{x} + \underbrace{O(h^3)}_{\text{estimate error - ignore}}$$

system dynamics  $\dot{x} = f(x)$

by chain rule  $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} = f'f$

$$x(t+h) = x(t) + hf + \frac{h^2}{2} ff'$$

or  $\Delta x = hf + \frac{h^2}{2} ff'$  (1)

Taylor series for system dynamics function

$$f(x+\delta) = f(x) + \delta f' + \underbrace{O(\delta^2)}_{\text{estimate error - ignore}}$$

let  $\delta$  be the midpoint of one Euler step

$$\delta = \frac{h}{2} f$$

$$f(x + \frac{h}{2} f) = f + \frac{h}{2} ff'$$

so

$$hf(x + \frac{h}{2} f) = hf + \frac{h^2}{2} ff' = \Delta x \text{ by eqn (1)}$$

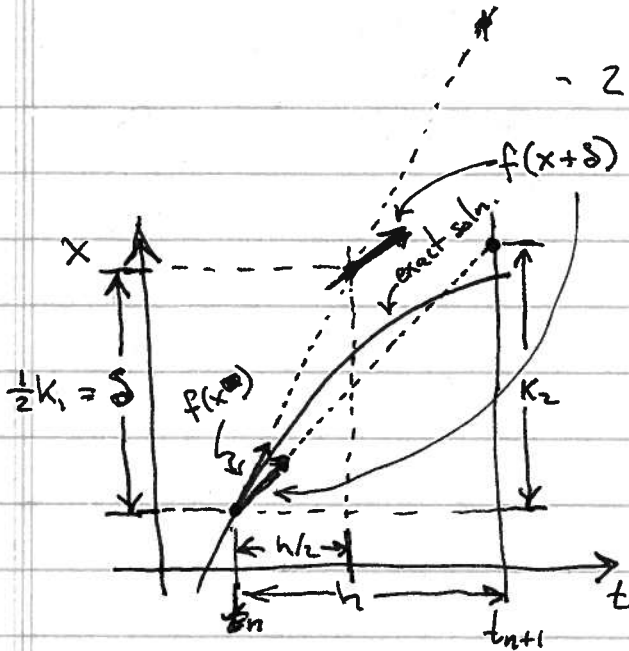
Thus,  $\Delta x = hf(x + \frac{h}{2} f)$

RK2:

$$k_1 = hf(x^{(n)})$$

$$k_2 = hf(x^{(n)} + \frac{1}{2} k_1) \quad (\text{this is } \Delta x)$$

$$x^{(n+1)} = x^{(n)} + k_2$$



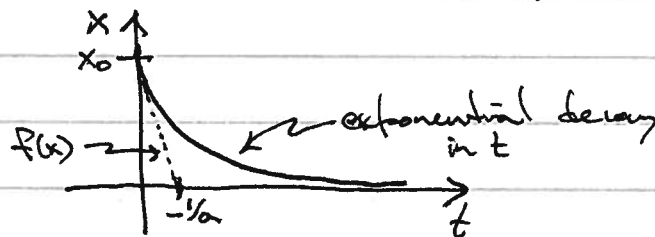
What are we doing with RK2 "Midpoint" method?

$\frac{1}{2}k_1$  is  $\delta$ , i.e. it is the change in  $x$  at the midpoint of an Euler step, using  $\dot{x} = f(x)$  as slope.

Now, find the slope at  $x + \delta = x + \frac{1}{2}k_1$ . It is  $f(x + \frac{1}{2}k_1)$ . Go back to start and take a full Euler step using this slope. This is  $\Delta x = k_2 = h f(x + \frac{1}{2}k_1)$ .

Example: Exponential Decay

$$\dot{x} = -ax \Rightarrow f(x) = -ax, \text{ soln. } x = x_0 e^{-at}$$



$$k_1 = hf(x) = -ahx$$

$$k_2 = hf(x + \frac{1}{2}k_1) = h[-a(x - \frac{ah}{2}x)]$$

$$= -ahx + \frac{a^2 h^2}{2}x$$

$$x^{(n+1)} = x^{(n)} - ahx^{(n)} + \frac{a^2 h^2}{2}x^{(n)}$$

$$x^{(n+1)} = (1 - ah + \frac{a^2 h^2}{2})x^{(n)}, \text{ change quadratic in } h$$

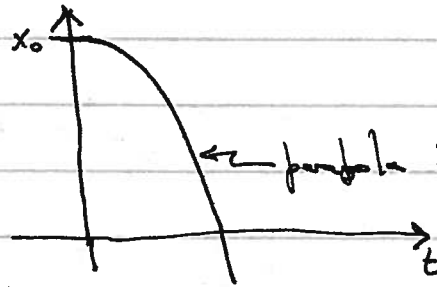
Euler would give

$$x^{(n+1)} = (1 - ah)x^{(n)}, \text{ change linear in } h$$

Example: Gravitational Acceleration

$$\underline{x} = \begin{bmatrix} x \\ v \end{bmatrix}, \quad f_v(\underline{x}) = -g, \quad f_x(\underline{x}) = v$$

$$\dot{\underline{x}} = \begin{bmatrix} v \\ -g \end{bmatrix} = f(\underline{x})$$



$$v = v_0 - gt$$

$$x = x_0 + v_0 t - \frac{1}{2}gt^2$$

$$\underline{k}_1 = hf(\underline{x}) = h \begin{bmatrix} v \\ -g \end{bmatrix}$$

$$\begin{aligned} \underline{k}_2 &= hf\left(\underline{x} + \frac{1}{2}\underline{k}_1\right) = \cancel{hf\left(\underline{x} + \frac{1}{2}h\underline{v}\right)} \\ &= hf\left(\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{2}hv \\ -\frac{1}{2}hg \end{bmatrix}\right) = hf\left(\begin{bmatrix} x + \frac{1}{2}hv \\ v - \frac{1}{2}hg \end{bmatrix}\right) \\ &= h \begin{bmatrix} v - \frac{1}{2}hg \\ -g \end{bmatrix} \end{aligned}$$

$$\underline{x}^{(n+1)} = \underline{x}^{(n)} + \begin{bmatrix} hv - \frac{1}{2}h^2g \\ -gh \end{bmatrix}$$

So:  $x^{(n+1)} = x^{(n)} + hv^{(n)} - \frac{1}{2}h^2g$ , change parabolic in  $h$   
 $v^{(n+1)} = v^{(n)} - gh$ , change linear in  $h$

Note that solution is exact, no matter what the time step