1. Radiometric Units

Photon Energy

Consider the equations and units of some familiar mechanics quantities from physics,

- A mass m is measured in kilograms [kg]
- Distance x is measured in meters [m]
- Time t is measured in seconds [s]
- Acceleration $a = x/t^2$ has units $\mathrm{m/s^2}$
- Force: $F = ma$ is measured in newtons, $[1\ \mathrm{N} = \mathrm{kg} \cdot \mathrm{m/s^2}]$
- Work, or Energy: $E = Fx$ (also $KE = \frac{1}{2} mv^2$) is measured in joules, $[1\ \mathrm{J} = \mathrm{N} \cdot \mathrm{m}]$

Light is an electromagnetic wave. It carries **energy**, which is transported by discrete **photons**. A reasonable mental model for now is that a ray of light is a stream of infinitesmal balls of energy, moving very fast. The energy in light is measured, as with any other energy, in joules.

Power

So that our rendering equations can be independent of the amount of time for the exposure, we'll consider the rate at which energy flows through a point. This rate called is **power** (a.k.a. **flux**) and has units of energy over time, which are watts $[1\mathrm{W} = \mathrm{J/s}]$. This is why appliances are rated in watts (and kilowatts) — to tell you how fast they consume energy. Power is denoted by the function $\Phi(\mathbb{A})$, measuring the amount of energy incident on the *entire* oriented surface $\mathbb{A}$.

A high-power (single-frequency) beam of visible light contains more photons arriving than a low-power beam of light. The photons can't arrive *faster* because they are all constrained to propagate at the same speed (i.e., the speed of light in that medium). The individual photons can't have more energy because photons have fixed energy determined by their frequency.

Irradiance and Radiosity

Power measures the rate of energy arrival over an entire surface. **Irradiance** is the power arriving *at a small patch of the surface* divided by the area measure of that patch. Its units are power per area measure, $\mathrm{W/m^2}$. When discussing the spatial rate at which power leaves the surface, the quantity is called **radiosity** and has the same units.

In the limit as the area of the considered patch shrinks, irradiance and radiosity describe how light arrives *at a point* and are denoted $E(X)$ and $B(X)$ respectively. Note that these quantities ignore the direction from which the light entered or left the point.

The Measure of a Set

To measure the light along a ray, we need to restrict the direction along which it is measured. This requires some new mathematical tools for describing sets of directions.

Consider the unit progression that I just followed in the preceeding sections. To restrict the measurement of light to a single instant, I divided energy by time [s], yielding power [W = J/s]. I then restricted the measurement to a single point on a surface by dividing by the area measure [m ^{2}] of that surface, yielding irradiance [$\mathrm{W/m^2}$]. What should I divide by to restrict the measurement to a direction?

An area is a set of 3D points that is locally two-dimensional (at any point on a surface, the local neighborhood is an edge or looks like a plane). Because it is a set, I use set formatting for an area variable: $\mathbb{A}$. The **measure** of an area is a scalar describing how much 2D space it would cover if flattened. It is denoted $||\mathbb{A}||$. In English, we use the word “area” interchangably for the set of points *and* for the scalar measurement of that set of points. The distinction is crucial in graphics, so I try to always say “area measure” when I'm referring to the scalar.

A planar **angle** is a set of points, just as an area is. The points comprising an angle are also on a 2D plane, but are further constrained to lie on the unit circle, $\mathbb{S}^1$ instead of anywhere in space. We're used to considering angles composed of contiguous points. However, under some definitions, any set of points on the unit circle forms an angle, although this rarely arises in graphics. When you depict an angle as an arc between two lines at their intersection, that arc really does comprise the points of the angle. Because they are sets of points, denote angles as sets, but using capital Greek letters, e.g., Θ.

Just as “area” can refer to a set or its measure, “angle” is casually applied to both the set of points anyone, regardless of nationality. So, user interfaces are probably best left in degrees.

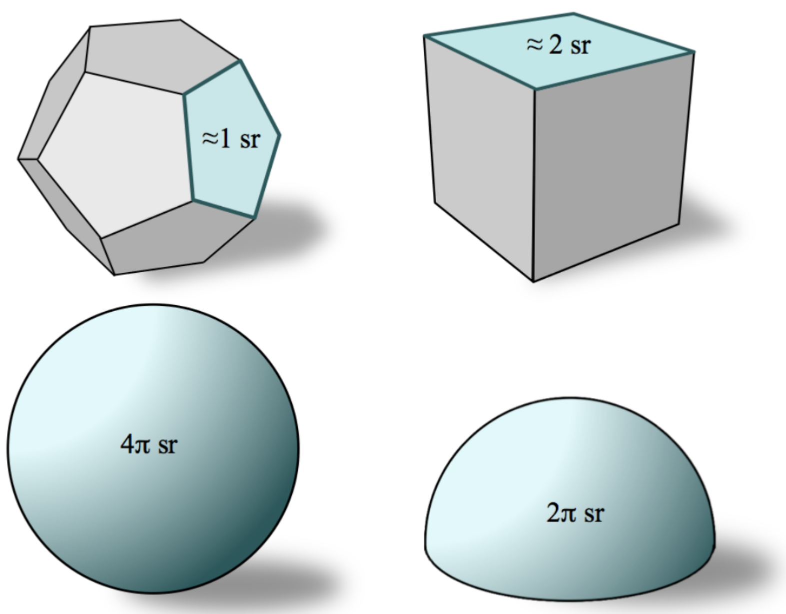
The measure of the entire unit circle is the circumference (i.e., length measure) of the unit circle,

$$||\mathbb{S}^1|| = 2\pi \text{ rad} \tag{1}$$

A direction is a point on the unit sphere, $\mathbb{S}^2$. A set of directions forms a region on the unit sphere called a **solid angle**. The directions can be contiguous, or disconnected. A common solid angle variable is Γ. Just as with an area or a planar angle, we can take the measure of a solid angle. In this case, the measure describes how much of the unit sphere is covered. $||\Gamma||$ is a scalar with units of **steradians**, where $1\text{sr} = 1 \text{ rad}^2$. The measure of the entire sphere is its surface area measure,

$$||\mathbb{S}^2|| = 4\pi \text{ sr} \tag{2}$$

The diagram below shows the solid angle measures of some simple regions (about the center of each shape) to help develop your intuition for the size of one steradian.



Measures of the solid angles of faces of some simple shapes.

We now have the answer to how to compute a measurement of irradiance over a limited set of directions: divide by the measure of the solid angle subtended.

Exercise **[Earth]:**

A visual summary of the major radiometric quantities follows, including some that I have not presented in this chapter. In the diagram, X is at the center of the horizontal cross, the photons are depicted as red balls, and the blue shapes simulate the measurement domain.

d

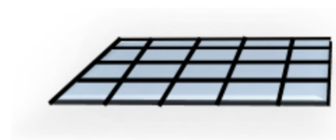
Length [m]

Size of a curve in $\mathbb{R}^3$

θ

Angle Measure [rad]

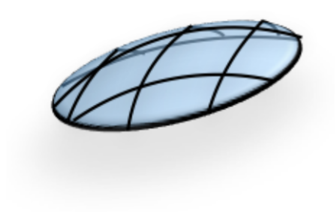
Size of an arc on the unit circle



$||\mathbb{A}||$

Area Measure [m^2]

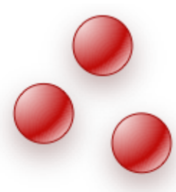
Size of a surface in $\mathbb{R}^3$



$||\Gamma||$

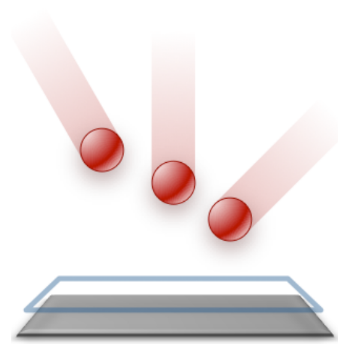
Solid Angle Measure [sr]

Size of a surface on the unit sphere, $\mathbb{S}^2$



Q

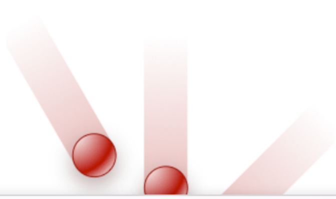
Energy [J]



$\Phi(t)$

Power [W]

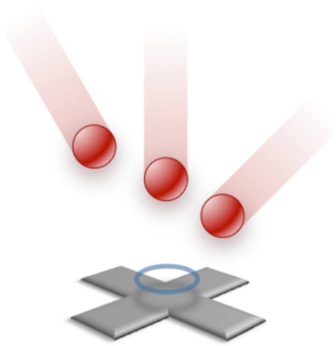
Energy transmission rate through a **surface** from **all** directions



sr]

Size of a surface on the unit sphere, $\mathbb{S}^2$

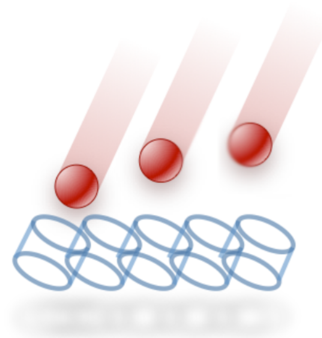
Energy transmission rate through a **surface** from **all** directions



$E(X)$

Irradiance $[\mathrm{W/m^2}]$

Incoming power at a **point** from **all** directions

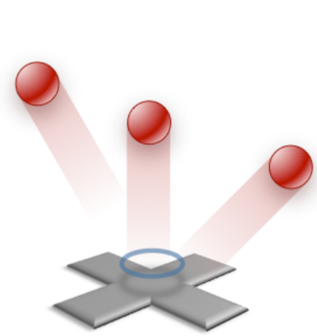


$I(\hat{\omega})$

Radiant Intensity

$[\mathrm{W/sr}]$

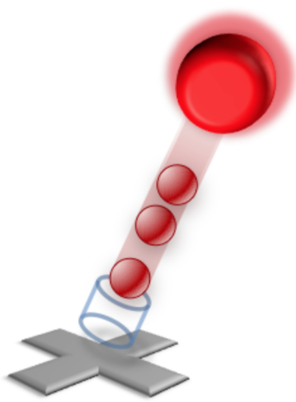
Incoming power at a **surface** from **one** direction



$B(X)$

Radiosity $[\mathrm{W/m^2}]$

Outgoing power at a **point** in **all** directions



$\beta(X, Y)$

Biradiance $[\mathrm{W/m^2}]$

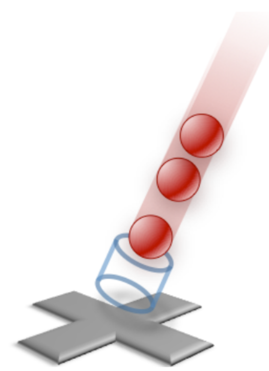
Solid-angle weighted radiance; net incident power at a **point** due to a point source at Y.

$M(X)$

Radiant Emittance

$[\mathrm{W/m^2}]$

Emitted power at a **point** in **all** directions



$L(X, \hat{\omega})$

Radiance $[\mathrm{W/(m^2 sr)}]$

Power at a **point** in **one** direction

Radiometric Units (Depicted) [rdmtryPic]

Lambert's Projected Area

I can measure how fast you are moving by dividing the time of the measurement period by the distance (the measure of a line segment) that you walked. However, say that I want to know your speed in the reference frame of the ground, but took a measurement while we were both on a moving train. I obviously have to add the train's velocity because I measured in a reference frame that distorted my measurement.

So, the reference frame in which we make a measurement affects its value and we must be careful to adjust for the reference frame when applying the measurement later during simulation. This leads to an important caveat about measuring radiance and irradiance:

> The observed measure of an area varies with the measure of the angle to the viewer.

For example, say that we measure the radiance incident from a direction $\hat{\omega}_\mathrm{i}$ that is 60 degrees off the vertical axis, at a point X just above the ground. This measurement is energy per time per area measure per solid angle measure. But at 60 degrees from vertical, the area on the ground only appears to be half as much as it really is. So, our measurement in space above the ground will lead us to think that there is twice as much energy incoming per time, area measure, and solid angle measure. The area factor of $1/2$ comes from the cosine 60 degrees. **Lambert's cosine law** states that the observed area measure of a surface falls with the cosine of the measure of the angle (from the vertical) at which it is observed [Lambert1760Photometrie] . It is also the case that the observed area measure falls with the square of distance, but this won't affect radiance measure because it is conserved along a ray.

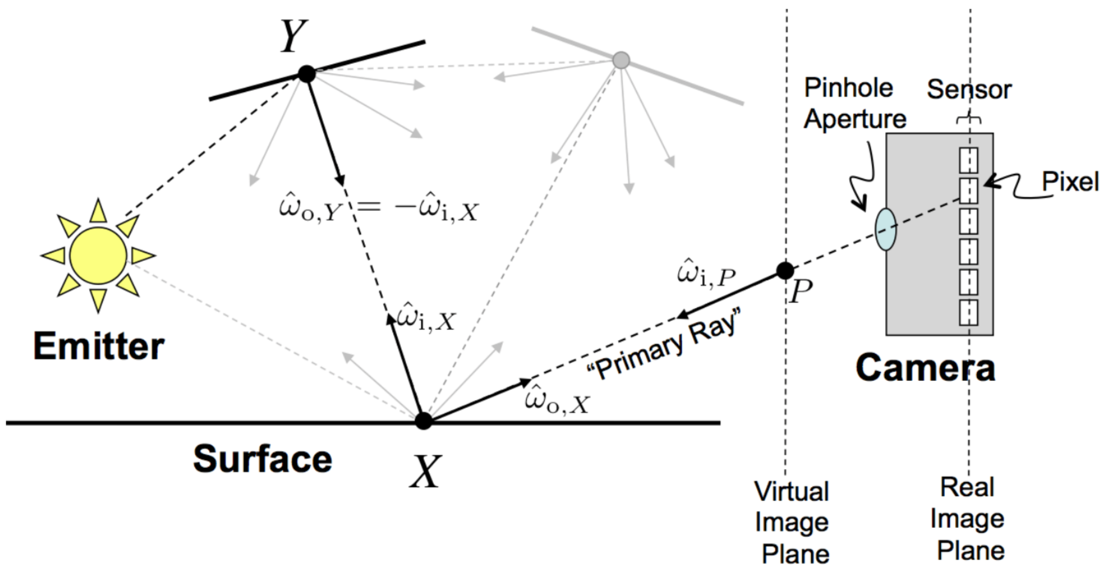
We need a way to compute the cosine compensation factor for radiance measurements so that they can be applied to compute actual energy deposited at a point, regardless of the angle of incidence. Fortunately, this is easily provided by vector operator called the dot product.

Vector Dot Product

The **dot product** (a.k.a. **inner product**) of 3D vectors $\vec{a} = (a_0, a_1, a_2)$ and $\vec{b} = (b_0, b_1, b_2)$ is the scalar quantity

$$\vec{a} \cdot \vec{b} = a_0 b_0 + a_1 b_1 + a_2 b_2 \tag{3}$$

Note that this has units that are the product of the units of $\vec{a}$ and $\vec{b}$. The dot product is defined in



$$L_o(X, \hat{\omega}_{o,X}) = L_e(X, \hat{\omega}_{o,X}) + \int L_i(X, \hat{\omega}_{i,X}) \ldots \square \ldots dY \tag{7}$$

where

$$\hat{\omega}_{i,X} = -\hat{\omega}_{o,Y} \tag{8}$$

You now know that the light field L carries radiance units. The empty box needs to hold factors that account for the amount of scattering and the measurement frame of L_i. The measurement frame is along $\hat{\omega}_{i,X}$. You also now know that for a surface with normal $\hat{n}_X$ at X, the compensation factor for measuring away from the normal is given by Lambert's law: $|\hat{\omega}_{i,X} \cdot \hat{n}|$.

Scattering depends on the material. There are many good models of materials. For now, let's abstract the scattering at X with a function of the incoming and outgoing directions (and normal) at the point, $f_{X,\hat{n}}(\hat{\omega}_{i,X}, \hat{\omega}_{o,X})$. This function is called the Bidirectional Scattering Distribution Function (BSDF).

Integrating with respect to the points from which light arrived is hard because we don't know which surfaces can be seen from X (and would force us to adjust for the projected area at those surfaces). Instead, let's just consider all possible *directions* from which light might arrive at X, i.e., over the unit sphere $\mathbb{S}^2$. Substituting these changes into the equation and dropping the point subscripts on $\hat{\omega}$ (all of which were X) gives the Rendering Equation:

$$L_o(X \in \mathbb{R}^3 \mathrm{m}, \hat{\omega}_o \in \mathbb{S}^2) \in \mathrm{W/(m^2 sr)} = \underbrace{L_e(X, \hat{\omega}_o)}_{\mathrm{W/(m^2 sr)}} + \int_{\mathbb{S}^2} \underbrace{L_i(X, \hat{\omega}_i)}_{\mathrm{W/(m^2 sr)}} \underbrace{f_{X,\hat{n}}(\hat{\omega}_i, \hat{\omega}_o)}_{\mathrm{sr}^{-1}} |\hat{\omega}_i \cdot \hat{n}| \underbrace{d\hat{\omega}_i}_{\mathrm{sr}} \tag{9}$$