3. $\mathbf{a} = [0, 2]$ $\mathbf{b} = [0, -2]$ $\mathbf{c} = [0.5, 2]$
$\mathbf{d} = [0.5, 2]$ $\mathbf{e} = [0.5, -3]$ $\mathbf{f} = [-2, 0]$
$\mathbf{g} = [-2, 1]$ $\mathbf{h} = [2.5, 2]$ $\mathbf{i} = [6, 1]$

4. (a) *The size of a vector in a diagram doesn't matter; we just need to draw it in the right place.* **False.** This is reversed; for vectors, size matters (meaning the length of the vector), position doesn't.

   (b) *The displacement expressed by a vector can be visualized as a sequence of axially aligned displacements.* **True.**

   (c) *These axially aligned displacements from the previous question must occur in order.* **False.** We can apply them in any order and get the same end result.

   (d) *The vector $[x, y]$ gives the displacement from the point (x, y) to the origin.* **False.** This is reversed; the vector $[x, y]$ gives the displacement from the origin to the point (x, y).

5. (a) $-\begin{bmatrix}3 & 7\end{bmatrix} = \begin{bmatrix}-3 & -7\end{bmatrix}$

   (b) $\left\|\begin{bmatrix}-12 & 5\end{bmatrix}\right\| = \sqrt{(-12)^2 + 5^2} = \sqrt{169} = 13$

   (c) $$\begin{aligned}\left\|\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}\right\| &= \sqrt{8^2 + (-3)^2 + (1/2)^2} = \sqrt{64 + 9 + (1/4)} \\ &= \sqrt{293/4} \approx 8.56\end{aligned}$$

   (d) $3\begin{bmatrix}4 & -7 & 0\end{bmatrix} = \begin{bmatrix}(3)(4) & (3)(-7) & (3)(0)\end{bmatrix} = \begin{bmatrix}12 & -21 & 0\end{bmatrix}$

   (e) $\begin{bmatrix}4 & 5\end{bmatrix}/2 = \begin{bmatrix}2 & 5/2\end{bmatrix}$

6. (a) $$\begin{aligned}\begin{bmatrix}12 & 5\end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix}12 & 5\end{bmatrix}}{\left\|\begin{bmatrix}12 & 5\end{bmatrix}\right\|} = \frac{\begin{bmatrix}12 & 5\end{bmatrix}}{13} = \begin{bmatrix}\frac{12}{13} & \frac{5}{13}\end{bmatrix} \\ &\approx \begin{bmatrix}0.923 & 0.385\end{bmatrix}\end{aligned}$$

   (b) $$\begin{aligned}\begin{bmatrix}0 & 743.632\end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix}0 & 743.632\end{bmatrix}}{\left\|\begin{bmatrix}0 & 743.632\end{bmatrix}\right\|} = \frac{\begin{bmatrix}0 & 743.632\end{bmatrix}}{\sqrt{0^2 + 743.632^2}} \\ &= \frac{\begin{bmatrix}0 & 743.632\end{bmatrix}}{743.632} = \begin{bmatrix}0 & 1\end{bmatrix}\end{aligned}$$

   (c) $$\begin{aligned}\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}}{\left\|\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}\right\|} \approx \frac{\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}}{8.56} \\ &\approx \begin{bmatrix}0.935 & -0.350 & 0.058\end{bmatrix}\end{aligned}$$

(d) $\begin{bmatrix}-12 & 3 & -4\end{bmatrix}_{\text{norm}} = \dfrac{\begin{bmatrix}-12 & 3 & -4\end{bmatrix}}{\left\|\begin{bmatrix}-12 & 3 & -4\end{bmatrix}\right\|} = \dfrac{\begin{bmatrix}-12 & 3 & -4\end{bmatrix}}{\sqrt{(-12)^2+3^2+(-4)^2}}$

$$= \frac{\begin{bmatrix}-12 & 3 & -4\end{bmatrix}}{13} = \begin{bmatrix}\frac{-12}{13} & \frac{3}{13} & \frac{-4}{13}\end{bmatrix}$$

(e) $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}_{\text{norm}} = \dfrac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}}{\left\|\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\right\|} = \dfrac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}}{\sqrt{1^2+1^2+1^2+1^2}}$

$$= \frac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}}{2} = \begin{bmatrix}0.5 & 0.5 & 0.5 & 0.5\end{bmatrix}$$

7. (a) $\begin{bmatrix}7 & -2 & -3\end{bmatrix} + \begin{bmatrix}6 & 6 & -4\end{bmatrix} = \begin{bmatrix}7+6 & -2+6 & -3+(-4)\end{bmatrix} = \begin{bmatrix}13 & 4 & -7\end{bmatrix}$

(b) $\begin{bmatrix}2 & 9 & -1\end{bmatrix} + \begin{bmatrix}-2 & -9 & 1\end{bmatrix} = \begin{bmatrix}2+(-2) & 9+(-9) & -1+1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$

(c) $\begin{bmatrix}3\\10\\7\end{bmatrix} - \begin{bmatrix}8\\-7\\4\end{bmatrix} = \begin{bmatrix}3-8\\10-(-7)\\7-4\end{bmatrix} = \begin{bmatrix}-5\\17\\3\end{bmatrix}$

(d) $\begin{bmatrix}4\\5\\-11\end{bmatrix} - \begin{bmatrix}-4\\-5\\11\end{bmatrix} = \begin{bmatrix}4-(-4)\\5-(-5)\\-11-11\end{bmatrix} = \begin{bmatrix}8\\10\\-22\end{bmatrix}$

(e) $3\begin{bmatrix}a\\b\\c\end{bmatrix} - 4\begin{bmatrix}2\\10\\-6\end{bmatrix} = \begin{bmatrix}3a\\3b\\3c\end{bmatrix} - \begin{bmatrix}8\\40\\-24\end{bmatrix} = \begin{bmatrix}3a-8\\3b-40\\3c+24\end{bmatrix}$

8. (a) $\text{distance}\left(\begin{bmatrix}10\\6\end{bmatrix}, \begin{bmatrix}-14\\30\end{bmatrix}\right) = \sqrt{(10-(-14))^2+(6-30)^2}$

$$= \sqrt{24^2+(-24)^2} = \sqrt{576+576}$$
$$= \sqrt{1152} \approx 33.94$$

(b) $\text{distance}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-12\\5\end{bmatrix}\right) = \sqrt{(0-(-12))^2+(0-5)^2}$

$$= \sqrt{12^2+(-5)^2} = \sqrt{144+25}$$
$$= \sqrt{169} = 13$$

(c) $\text{distance}\left(\begin{bmatrix}3\\10\\7\end{bmatrix}, \begin{bmatrix}8\\-7\\4\end{bmatrix}\right) = \sqrt{(3-8)^2+(10-(-7))^2+(7-4)^2}$

$$= \sqrt{(-5)^2+17^2+3^2} = \sqrt{25+289+9}$$
$$= \sqrt{323} \approx 17.97$$

(d) $\text{distance}\left(\begin{bmatrix}-2\\-4\\9\end{bmatrix}, \begin{bmatrix}6\\-7\\9.5\end{bmatrix}\right) = \sqrt{(6-(-2))^2+(-7-(-4))^2+(9.5-9)^2}$

$$= \sqrt{8^2+(-3)^2+(0.5)^2} = \sqrt{64+9+0.25}$$
$$= \sqrt{73.25} \approx 8.56$$

(e) $\text{distance}\left(\begin{bmatrix}4\\-4\\-4\\4\end{bmatrix}, \begin{bmatrix}-6\\6\\6\\-6\end{bmatrix}\right) = \sqrt{(-6-4)^2+(6-(-4))^2+(6-(-4))^2+(-6-4)^2}$

$$= \sqrt{(-10)^2+(10)^2+(10)^2+(-10)^2}$$
$$= \sqrt{100+100+100+100}$$
$$= \sqrt{400} = 20$$

9. (a) $\begin{bmatrix}2\\6\end{bmatrix} \cdot \begin{bmatrix}-3\\8\end{bmatrix} = (2)(-3)+(6)(8) = -6+48 = 42$

(b) $-7\begin{bmatrix}1 & 2\end{bmatrix} \cdot \begin{bmatrix}11 & -4\end{bmatrix} = \begin{bmatrix}-7 & -14\end{bmatrix} \cdot \begin{bmatrix}11 & -4\end{bmatrix}$

$$= (-7)(11)+(-14)(-4)$$
$$= -21$$

(c) $10 + \begin{bmatrix}-5\\1\\3\end{bmatrix} \cdot \begin{bmatrix}4\\-13\\9\end{bmatrix} = 10 + ((-5)(4)+(1)(-13)+(3)(9))$

$$= 10 + (-20+(-13)+27)$$
$$= 10 + (-6) = 4$$

(d) $3\begin{bmatrix}-2\\0\\4\end{bmatrix} \cdot \left(\begin{bmatrix}8\\-2\\3/2\end{bmatrix} + \begin{bmatrix}0\\9\\7\end{bmatrix}\right) = \begin{bmatrix}-6\\0\\12\end{bmatrix} \cdot \begin{bmatrix}8\\7\\17/2\end{bmatrix}$

$$= (-6)(8)+(0)(7)+(12)(17/2) = 54$$

10. $\mathbf{v}_{\parallel} = \hat{\mathbf{n}}\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|^2} = \hat{\mathbf{n}}\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{1} = \hat{\mathbf{n}}\left(\mathbf{v}\cdot\hat{\mathbf{n}}\right)$

$$= \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\left(\begin{bmatrix}4\\3\\-1\end{bmatrix} \cdot \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\right) = \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\left(2\sqrt{2}+\tfrac{3\sqrt{2}}{2}+0\right)$$

$$= \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\tfrac{7\sqrt{2}}{2} = \begin{bmatrix}7/2\\7/2\\0\end{bmatrix}$$

$$\mathbf{p}_{\text{BackUpperLeft}} = \begin{bmatrix} c_x - r_x \\ c_y + r_y \\ c_z - r_z \end{bmatrix}, \qquad \mathbf{p}_{\text{BackUpperRight}} = \begin{bmatrix} c_x + r_x \\ c_y + r_y \\ c_z - r_z \end{bmatrix},$$

$$\mathbf{p}_{\text{BackLowerLeft}} = \begin{bmatrix} c_x - r_x \\ c_y - r_y \\ c_z - r_z \end{bmatrix}, \qquad \mathbf{p}_{\text{BackLowerRight}} = \begin{bmatrix} c_x + r_x \\ c_y - r_y \\ c_z - r_z \end{bmatrix}.$$

20. (a) Use the sign of the dot product between $\mathbf{v}$ and $\mathbf{x} - \mathbf{p}$ to determine whether the point $\mathbf{x}$ is in front of or behind the NPC. This follows from the geometric interpretation of the dot product,

$$\mathbf{v} \cdot (\mathbf{x} - \mathbf{p}) = \|\mathbf{v}\|\|\mathbf{x} - \mathbf{p}\| \cos\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{x} - \mathbf{p}$.

Both $\|\mathbf{v}\|$ and $\|\mathbf{x} - \mathbf{p}\|$ are always positive, leaving the sign of the dot product entirely up to the value of $\cos\theta$. If $\cos\theta > 0$ then θ is less than 90° and $\mathbf{x}$ is *in front of* the NPC. Similarly, if $\cos\theta < 0$ then θ is greater than 90° and $\mathbf{x}$ is *behind* the NPC.

The special case of $\mathbf{v} \cdot (\mathbf{x} - \mathbf{p}) = 0$ means that $\mathbf{x}$ lies either directly to the left or right of the NPC. If this case does not need to be handled explicitly, it can arbitrarily be assigned to mean either in front of or behind.

(b) (1) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} = (5)(3) + (-2)(-4) = 23$$

(2) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 6 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (5)(4) + (-2)(2) = 16$$

(3) $\mathbf{x}$ is behind the NPC.

$$\begin{aligned} \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} -6 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -4 \end{bmatrix} &= (5)(-3) + (-2)(-4) \\ &= -7 \end{aligned}$$

(4) $\mathbf{x}$ is behind the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} -4 \\ 7 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (5)(-1) + (-2)(3) = -11$$

(5) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 1 \end{bmatrix} = (5)(8) + (-2)(1) = 38$$

(6) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} -3 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -4 \end{bmatrix} = (5)(0) + (-2)(-4) = 8$$

(7) $\mathbf{x}$ can be either in front of or behind the NPC, depending on how we've decided to handle this special case.

$$\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-6\\-3.5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)=\begin{bmatrix}5\\-2\end{bmatrix}\cdot\begin{bmatrix}-3\\-7.5\end{bmatrix}=(5)(-3)+(-2)(-7.5)=0$$

21. (a) To determine whether the point $\mathbf{x}$ is visible to the NPC, compare $\cos\theta$ to $\cos(\phi/2)$. If $\cos\theta \geq \cos(\phi/2)$, then $\mathbf{x}$ is visible to the NPC.

The value of $\cos(\phi/2)$ can be obtained from the FOV angle. To get $\cos\theta$ use the dot product

$$\cos\theta = \frac{\mathbf{v}\cdot(\mathbf{x}-\mathbf{p})}{\|\mathbf{v}\|\|\mathbf{x}-\mathbf{p}\|}.$$

(b) The NPC's FOV is 90°, so the value we are interested in is $\cos(45°) \approx 0.707$.

(1) $\mathbf{x}$ is visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{23}{(\sqrt{29})(\sqrt{(25)})}\approx 0.854 \geq 0.707$$

(2) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}1\\6\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}1\\6\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{16}{(\sqrt{29})(\sqrt{20})}\approx 0.664 < 0.707$$

(3) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-6\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-6\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{-7}{(\sqrt{29})(\sqrt{25})}$$

$$\approx -0.260 < 0.707$$

(4) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-4\\7\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-4\\7\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{-11}{(\sqrt{29})(\sqrt{10})}\approx -0.646 < 0.707$$

(5) $\mathbf{x}$ is visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}5\\5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}5\\5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{38}{(\sqrt{29})(\sqrt{65})}\approx 0.875 \geq 0.707$$

(6) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-3\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-3\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|} = \frac{8}{(\sqrt{29})(\sqrt{16})} \approx 0.371 < 0.707$$

(7) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-6\\-3.5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-6\\-3.5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|} = \frac{0}{(\sqrt{29})(\sqrt{65.25})} = 0 < 0.707$$

(c) The NPC can see a distance of only 7 units, so only those points that are both within the FOV and within this distance will be visible.

(1) $\mathbf{x}$ is visible to the NPC.

$$\left\|\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\| = \left\|\begin{bmatrix}3\\-4\end{bmatrix}\right\| = \sqrt{25} = 5 < 7$$

(2) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(3) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(4) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(5) $\mathbf{x}$ is not visible to the NPC.

$$\left\|\begin{bmatrix}5\\5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\| = \left\|\begin{bmatrix}8\\1\end{bmatrix}\right\| = \sqrt{65} \approx 8.062 > 7$$

(6) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(7) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

22. (a) Let $\mathbf{v}_{ab} = \mathbf{b} - \mathbf{a}$ and $\mathbf{v}_{bc} = \mathbf{c} - \mathbf{b}$. Since the three points lie in the xz-plane, the two vectors also lie in the xz-plane and we have

$$\mathbf{v}_{ab} = \begin{bmatrix}x_{ab}\\0\\z_{ab}\end{bmatrix}, \qquad \mathbf{v}_{bc} = \begin{bmatrix}x_{bc}\\0\\z_{bc}\end{bmatrix}.$$

Taking the cross product of the vectors in the order that the points are traversed gives.

$$\mathbf{v}_{ab} \times \mathbf{v}_{bc} = \begin{bmatrix}0\\x_{bc}z_{ab} - x_{ab}z_{bc}\\0\end{bmatrix}$$

The sign of $x_{bc}z_{ab} - x_{ab}z_{bc}$ can then be used to determine the NPC's turning direction. Because we are working in a left-handed coordinate system, if the value is negative, the NPC is turning counterclockwise; if it's positive he's turning clockwise. The special case of 0 signifies that the NPC is either walking forward in a straight line or walks forward and then back along the same line.

3.
$$\begin{array}{rcccl} \mathbf{AB} & = & (4\times3)(3\times3) & = & 4\times3 \\ \mathbf{BB} & = & (3\times3)(3\times3) & = & 3\times3 \\ \mathbf{CC} & = & (2\times2)(2\times2) & = & 2\times2 \\ \mathbf{EB} & = & (1\times3)(3\times3) & = & 1\times3 \\ \mathbf{FE} & = & (4\times1)(1\times3) & = & 4\times3 \\ \mathbf{GA} & = & (1\times4)(4\times3) & = & 1\times3 \\ \mathbf{HE} & = & (3\times1)(1\times3) & = & 3\times3 \end{array} \qquad \begin{array}{rcccl} \mathbf{AH} & = & (4\times3)(3\times1) & = & 4\times1 \\ \mathbf{BH} & = & (3\times3)(3\times1) & = & 3\times1 \\ \mathbf{DC} & = & (5\times2)(2\times2) & = & 5\times2 \\ \mathbf{EH} & = & (1\times3)(3\times1) & = & 1\times1 \\ \mathbf{FG} & = & (4\times1)(1\times4) & = & 4\times4 \\ \mathbf{GF} & = & (1\times4)(4\times1) & = & 1\times1 \\ \mathbf{HG} & = & (3\times1)(1\times4) & = & 3\times4 \end{array}$$

4. (a)
$$\begin{bmatrix}1 & -2\\ 5 & 0\end{bmatrix}\begin{bmatrix}-3 & 7\\ 4 & 1/3\end{bmatrix} = \begin{bmatrix}(1)(-3)+(-2)(4) & (1)(7)+(-2)(1/3)\\ (5)(-3)+(0)(4) & (5)(7)+(0)(1/3)\end{bmatrix}$$
$$= \begin{bmatrix}-3+(-8) & 7+(-2/3)\\ -15+0 & 35+0\end{bmatrix} = \begin{bmatrix}-11 & 19/3\\ -15 & 35\end{bmatrix}$$

(b) Not possible; cannot multiply a 2 × 2 matrix by a 1 × 2 vector on the right.

(c)
$$\begin{bmatrix}3 & -1 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 3\\ 5 & 7 & -6\\ 1 & -4 & 2\end{bmatrix}$$
$$= \begin{bmatrix}(3)(-2)+(-1)(5)+(4)(1) & (3)(0)+(-1)(7)+(4)(-4) & (3)(3)+(-1)(-6)+(4)(2)\end{bmatrix}$$
$$= \begin{bmatrix}-6+(-5)+4 & 0+(-7)+(-16) & 9+6+8\end{bmatrix} = \begin{bmatrix}-7 & -23 & 23\end{bmatrix}$$

(d)
$$\begin{bmatrix}x & y & z & w\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}x & y & z & w\end{bmatrix}$$

(e) Not possible; cannot multiply a 1 × 4 vector by a 2 × 1 vector.

(f)
$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix} = \begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}$$

(g)
$$\begin{bmatrix}3 & 3\end{bmatrix}\begin{bmatrix}6 & -7\\ -4 & 5\end{bmatrix} = \begin{bmatrix}(3)(6)+(3)(-4) & (3)(-7)+(3)(5)\end{bmatrix}$$
$$= \begin{bmatrix}18+(-12) & -21+15\end{bmatrix} = \begin{bmatrix}6 & -6\end{bmatrix}$$

(h) Not possible; cannot multiply a 3 × 3 matrix by a 2 × 3 matrix on the right.

5. (a)
$$\begin{bmatrix}5 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}(5)(1)+(-1)(0)+(2)(0) & (5)(0)+(-1)(1)+(2)(0) & (5)(0)+(-1)(0)+(2)(1)\end{bmatrix}$$
$$= \begin{bmatrix}5 & -1 & 2\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\ -1\\ 2\end{bmatrix} = \begin{bmatrix}(1)(5)+(0)(-1)+(0)(2)\\ (0)(5)+(1)(-1)+(0)(2)\\ (0)(5)+(0)(-1)+(1)(2)\end{bmatrix} = \begin{bmatrix}5\\ -1\\ 2\end{bmatrix}$$

(b) $\begin{bmatrix}5 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & 5 & -3\\1 & 7 & 1\\-2 & -1 & 4\end{bmatrix}$

$$= \begin{bmatrix}(5)(2)+(-1)(1)+(2)(-2) & (5)(5)+(-1)(7)+(2)(-1) & (5)(-3)+(-1)(1)+(2)(4)\end{bmatrix}$$
$$= \begin{bmatrix}10+(-1)+(-4) & 25+(-7)+(-2) & -15+(-1)+8\end{bmatrix} = \begin{bmatrix}5 & 16 & -8\end{bmatrix}$$
$$\begin{bmatrix}2 & 5 & -3\\1 & 7 & 1\\-2 & -1 & 4\end{bmatrix}\begin{bmatrix}5\\-1\\2\end{bmatrix} = \begin{bmatrix}(2)(5)+(5)(-1)+(-3)(2)\\(1)(5)+(7)(-1)+(1)(2)\\(-2)(5)+(-1)(-1)+(4)(2)\end{bmatrix} = \begin{bmatrix}10+(-5)+(-6)\\5+(-7)+2\\-10+1+8\end{bmatrix} = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$$

(c) $\begin{bmatrix}5 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 7 & 2\\7 & 0 & -3\\2 & -3 & -1\end{bmatrix}$

$$= \begin{bmatrix}(5)(1)+(-1)(7)+(2)(2) & (5)(7)+(-1)(0)+(2)(-3) & (5)(2)+(-1)(-3)+(2)(-1)\end{bmatrix}$$
$$= \begin{bmatrix}5+(-7)+4 & 35+0+(-6) & 10+3+(-2)\end{bmatrix} = \begin{bmatrix}2 & 29 & 11\end{bmatrix}$$
$$\begin{bmatrix}1 & 7 & 2\\7 & 0 & -3\\2 & -3 & -1\end{bmatrix}\begin{bmatrix}5\\-1\\2\end{bmatrix} = \begin{bmatrix}(1)(5)+(7)(-1)+(2)(2)\\(7)(5)+(0)(-1)+(-3)(2)\\(2)(5)+(-3)(-1)+(-1)(2)\end{bmatrix} = \begin{bmatrix}5+(-7)+4\\35+0+(-6)\\10+3+(-2)\end{bmatrix} = \begin{bmatrix}2\\29\\11\end{bmatrix}$$

(d) $\begin{bmatrix}5 & -1 & 2\end{bmatrix}\begin{bmatrix}0 & -4 & 3\\4 & 0 & -1\\-3 & 1 & 0\end{bmatrix}$

$$= \begin{bmatrix}(5)(0)+(-1)(4)+(2)(-3) & (5)(-4)+(-1)(0)+(2)(1) & (5)(3)+(-1)(-1)+(2)(0)\end{bmatrix}$$
$$= \begin{bmatrix}0+(-4)+(-6) & (-20)+0+2 & 15+1+0\end{bmatrix} = \begin{bmatrix}-10 & -18 & 16\end{bmatrix}$$
$$\begin{bmatrix}0 & -4 & 3\\4 & 0 & -1\\-3 & 1 & 0\end{bmatrix}\begin{bmatrix}5\\-1\\2\end{bmatrix} = \begin{bmatrix}(0)(5)+(-4)(-1)+(3)(2)\\(4)(5)+(0)(-1)+(-1)(2)\\(-3)(5)+(1)(-1)+(0)(2)\end{bmatrix} = \begin{bmatrix}0+4+6\\20+0+(-2)\\-15+(-1)+0\end{bmatrix} = \begin{bmatrix}10\\18\\-16\end{bmatrix}$$

6. (a) $\left((\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}$

(b) $(\mathbf{BA}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{CD}^{\mathrm{T}}) = \left((\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{B})^{\mathrm{T}}\right)(\mathbf{CD}^{\mathrm{T}}) = (\mathbf{AB}^{\mathrm{T}})(\mathbf{CD}^{\mathrm{T}}) = \mathbf{AB}^{\mathrm{T}}\mathbf{CD}^{\mathrm{T}}$

(c) $\left((\mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}})(\mathbf{AB})^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\left((\mathbf{AB})^{\mathrm{T}}\right)^{\mathrm{T}}(\mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}})^{\mathrm{T}}\right) = (\mathbf{AB})\left((\mathbf{C}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{D}^{\mathrm{T}})^{\mathrm{T}}\right)$

$= (\mathbf{AB})(\mathbf{CD}) = \mathbf{ABCD}$

(d) $\left((\mathbf{AB})^{\mathrm{T}}(\mathbf{CDE})^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\left((\mathbf{CDE})^{\mathrm{T}}\right)^{\mathrm{T}}\left((\mathbf{AB})^{\mathrm{T}}\right)^{\mathrm{T}}\right) = (\mathbf{CDE})(\mathbf{AB})$

$= \mathbf{CDEAB}$

7. For each of the matrices $\mathbf{M}$, interpret the rows of $\mathbf{M}$ as basis vectors after transformation.

(a) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[0, -1]$ and $[1, 0]$, respectively. Thus, $\mathbf{M}$ performs a 90° clockwise rotation.

(b) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$, respectively. Thus, $\mathbf{M}$ performs a 45° counterclockwise rotation.

(c) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[2, 0]$ and $[0, 2]$, respectively. Thus, $\mathbf{M}$ performs a *uniform* scale, scaling both the x and y dimensions by 2.

(d) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[4, 0]$ and $[0, 7]$, respectively. Thus, $\mathbf{M}$ performs a *nonuniform* scale, scaling the x dimension by 4 and the y dimension by 7.

(e) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[-1, 0]$ and $[0, 1]$, respectively. Thus, $\mathbf{M}$ performs a reflection across the y axis, negating x values and leaving y values untouched.

(f) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[0, -2]$ and $[2, 0]$, respectively. Thus, $\mathbf{M}$ is performing a combination of transformations: it is rotating clockwise by 90° and scaling both dimensions uniformly by 2. This can be confirmed by multiplying the appropriate matrices from parts (a) and (c), which perform these transformations individually:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

8. $\mathbf{M} = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$ This matrix is skew symmetric, as desired, since $\mathbf{M}^{\mathrm{T}} = -\mathbf{M}$.

9. (a) 3 (b) 1 (c) 4 (d) 2

10. The result vector element w_i is the product of the ith row of $\mathbf{M}$ multiplied by the column vector $\mathbf{v}$. To have $w_i = v_i - v_{i-1}$, the ith row of $\mathbf{M}$ needs to capture the ith element of $\mathbf{v}$, as well as the negative of the $(i-1)$th element, but exclude all others. This means that

$$m_{ij} = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

11. The result vector element w_i is the product of the ith row of $\mathbf{N}$ multiplied by the column vector $\mathbf{v}$. To have $w_i = \sum_{j=1}^{i} v_j$, the ith row of $\mathbf{N}$ needs to capture all elements of $\mathbf{v}$ up to and including the ith element, but exclude all others. This means that

$$n_{ij} = \begin{cases} 1 & \text{if } j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

12. (a) Note that the structure of $\mathbf{M}$ causes the ith row of $\mathbf{MN}$ to be equivalent to the difference between the ith and $(i-1)$th rows of $\mathbf{N}$.

    (b) Note that the structure of $\mathbf{N}$ causes the ith row of $\mathbf{NM}$ to be equivalent to the sum of the first i rows of $\mathbf{M}$.

    (c) $$\mathbf{MN} = \mathbf{NM} = \mathbf{I}_{10\times 10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

B.5 Chapter 5

(Page 159.)

1. Yes, *any* matrix expresses a linear transformation. Furthermore, because all linear transformations are also affine transformations, the transform is also an affine transformation. (There just isn't any translation in the affine transform, or equivalently, the translation portion is zero.)

2. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-22^\circ) & \sin(-22^\circ) \\ 0 & -\sin(-22^\circ) & \cos(-22^\circ) \end{bmatrix} = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.927 & -0.375 \\ 0.000 & 0.375 & 0.927 \end{bmatrix}$$

(f) $r = \sqrt{x^2+y^2+z^2} = \sqrt{3^2+4^2+12^2} = \sqrt{9+16+144} = 13$
$h = \arctan(x/z) = \arctan(3/12) = \arctan(1/4) = 14.04^o$
$p = \arcsin(-y/r) = \arcsin(-4/13) = -17.92^o$
so $(r,h,p) = (13, 14.04^o, -17.92^o)$

11. (a) A sphere with radius r_0.
    (b) A vertical plane, obtained by rotating the plane $x = 0$ clockwise about the y axis by the angle h_0.
    (c) A "right circular conical surface" (two vertical circular cones meeting tip-to-tip at the origin). The interior angle of the cone is $2p_0$.
12. She was at the north pole, so the bear was white.[1]

B.8 Chapter 8

(Page 291.)

1. (a) 5 (b) 3 (c) 6 (d) 1 (e) 2 (f) 4
2. (a) 3. Yes, they are canonical Euler angles.
   (b) 4. Yes, they are canonical Euler angles.
   (c) 5. No, this orientation is in Gimbal lock, and in the canonical set, bank should be zero.
   (d) 1. Yes, they are canonical Euler angles.
   (e) 2. Yes, they are canonical Euler angles.
   (f) 3. No, the pitch angle is outside the legal range.
   (g) 5. Yes, they are canonical Euler angles.
   (h) 2. No, the pitch angle is outside the legal range.
   (i) 6. Yes, they are canonical Euler angles.
3. (a) $$\begin{bmatrix} \cos(30^o/2) \\ \begin{pmatrix} 1\cdot\sin(30^o/2) \\ 0\cdot\sin(30^o/2) \\ 0\cdot\sin(30^o/2) \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0.966 \\ \begin{pmatrix} .259 \\ 0.000 \\ 0.000 \end{pmatrix} \end{bmatrix}$$
   (b) *All* rotation quaternions have a magnitude of 1!

[1]It was *polar* bear. Get it?! Polar!