Review the Discrete-Time Fourier Transform (DTFT)

The DTFT of a discrete time signal x(n) is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad (*)$$

If $X(e^{j\omega})$ is known, x(n) can be found using the inverse relation:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad (**)$$

Necessary and sufficient condition for uniform convergence of $(*)$ is:

$$\sum_{n=-\infty}^{\infty} |\, x(n)\,| < \infty \qquad (***)$$

When condition $(***)$ is <u>no</u>t satisfied, there are cases where the summation of $(*)$ may still converge in a non-uniform sense and where x(n) and $X(e^{j\omega})$ are related by equation $(**)$.

For example, if x(n) does not satisfy condition $(***)$ but does satisfy the following condition:

$$\sum_{n=-\infty}^{\infty} |\, x(n)\,|^2 < \infty$$

then the summation of $(*)$ converges in the <u>mean-squared sense</u> to a frequency function $X(e^{j\omega})$. An example of this case is the ideal low-pass filter whose unit sample response is given by

$$h_{lp}(n) = \frac{\sin(\omega_c n)}{\pi n}, \quad n \neq 0$$

$$= \frac{\omega_c}{\pi}, \quad n = 0$$

Example 2.18 - Square-Summability for the Ideal Lowpass Filter (p. 51)

If we define

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} h_{lp}(n) e^{-j\omega n}$$

then

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} | H_{lp}(e^{j\omega}) - H_M(e^{j\omega}) |^2 \, d\omega = 0$$

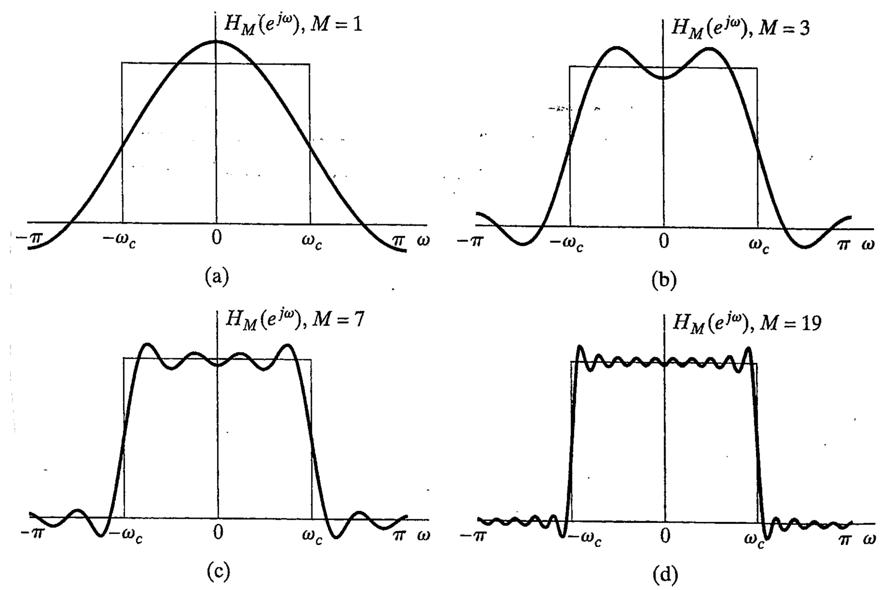


Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

There are some signals which are neither absolutely summable nor square summable but for which a Fourier Transform representation is still useful. The following example demonstrates such a case.

Example 2.20 - Fourier Transform of Complex Exponential Sequences

Start with the following $X(e^{j\omega})$ which consists of a periodic impulse train and use $(**)$ to find x(n):

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi r) \qquad (2.143)$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi r)\right]e^{j\omega n}\,d\omega$$

$$= \int_{-\pi}^{\pi}\left[\sum_{r=-\infty}^{\infty} \delta(\omega - \omega_o + 2\pi r)\right]e^{j\omega n}\,d\omega$$

Since the above integrand involves impulses spaced by 2π, there is exactly one impulse over $-\pi < \omega < \pi$ which is the range of the integral. This impulse function is located at $\omega = \omega_0 - 2\pi r$ for some integer r. The value of the integral is therefore

$e^{j(\omega_o - 2\pi r)n} = e^{j\omega_o n}$ = complex exponential signal

<u>Another example</u>: $\cos(\omega_o n + \phi)$

$$\cos(\omega_o n + \phi) = \frac{e^{j(\omega_o n+\varphi)} + e^{-j(\omega_o n+\varphi)}}{2} = \frac{e^{j\varphi}e^{j\omega_0 n}}{2} + \frac{e^{-j\varphi}e^{-j\omega_0 n}}{2}$$

Applying above results, we can obtain $X(e^{j\omega})$ for $x(n) = \cos(\omega_o n + \phi)$ as

$$X(e^{j\omega}) = \frac{e^{j\phi}}{2}\sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi r) + \frac{e^{-j\phi}}{2}\sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + \omega_o + 2\pi r)$$

$$= \sum_{r=-\infty}^{\infty}\left[\pi e^{j\phi}\delta(\omega - \omega_o + 2\pi r) + \pi e^{-j\phi}\delta(\omega + \omega_o + 2\pi r)\right]$$

Table 2.3 below shows more Fourier Transform pairs)

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{J}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{J}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Conjugate symmetric part of x(n): $x_e(n) = \dfrac{x(n) + x^*(-n)}{2} = x_e^*(-n)$

Conjugate anti-symmetric part of x(n): $x_o(n) = \dfrac{x(n) - x^*(-n)}{2} = -x_o^*(-n)$

For the special when x(n) is real: $x_e(n) = \dfrac{x(n) + x(-n)}{2}$ = even part of x(n)

$x_o(n) = \dfrac{x(n) - x(-n)}{2}$ = odd part of x(n)

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Section 4.4 - Discrete-Time Processing of Continuous-Time Signals

Consider the system structure shown in Figure 4.11 on p. 153:

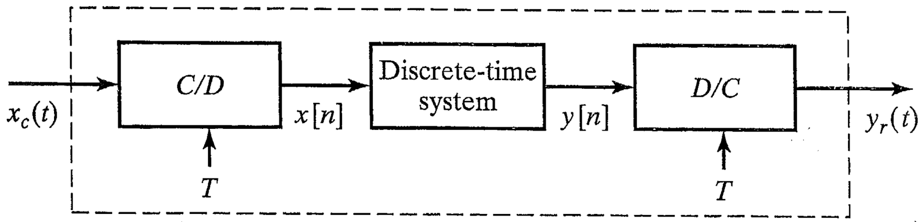


Figure 4.10 Discrete-time processing of continuous-time signals.

An example based on this structure:

Example 4.4 - Discrete-Time Implementation of an Ideal Continuous-Time Bandlimited Differentiator

Assume that we want the overall effect of the above system to be an ideal differentiator of the analog input, i.e., we want

$$y_r(t) = \frac{d}{dt}\left[x_c(t)\right]$$

In the s-domain this corresponds to the system operation $Y_r(s) = sX_c(s)$, which can be expressed in the analog frequency domain as

$$H_c(j\Omega) = j\Omega$$

To obtain a discrete time implementation, consider a band-limited version of the above. (Assume that the input in bandlimited and that it is sampled at a rate that exceeds the Nyquist rate.)

$H_{ef}(j\Omega) = j\Omega$ for $|\Omega| < \frac{\pi}{T}$

and

$H_{ef}(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$

The corresponding digital filter can be obtained by setting

$H(e^{j\omega}) = \frac{j\omega}{T}$ for $|\omega| < \pi$

(As with all digital filters, $X(e^{j\omega})$ will be periodic in ω with period = 2π.)

The unit sample response corresponding to $H(e^{j\omega})$ can be found using:

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{j\omega}{T} e^{j\omega n}\,d\omega \qquad (****)$$

This integral can be evaluated using "integration by parts":

Let $u = \omega$, then $du = d\omega$

and let $v = e^{j\omega n}$, so that $dv = jne^{j\omega n}d\omega$

Therefore, the right side of Equation $(****)$ can be written as

$$\frac{1}{2\pi nT}\int_{-\pi}^{\pi} u\,dv \qquad (*****)$$

Using integration by parts to perform the integral:

$$\int_{-\pi}^{\pi} u\,dv = uv\,|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} v\,du$$

$$= \omega e^{j\omega n}\,|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{j\omega n} d\omega$$

$$= \pi e^{j\pi n} - (-\pi e^{-j\pi n}) - \left.\frac{e^{j\omega n}}{jn}\right|_{-\pi}^{\pi}, \qquad n \neq 0$$

$$= \pi[e^{j\pi n} + e^{-j\pi n}] - \left[\frac{e^{j\omega n} - e^{-j\omega n}}{jn}\right], \qquad n \neq 0$$

$$= 2\pi\cos(\pi n) - \left[\frac{2}{n}\right]\sin(\pi n), \qquad n \neq 0$$

Therefore, Equation $(*****)$ becomes

$$h(n) = \frac{1}{2\pi nT}\left[2\pi\cos(\pi n) - \left[\frac{2}{n}\right]\sin(\pi n)\right], \qquad n \neq 0$$

$$= \frac{1}{\pi n^2 T}\left[\pi n\cos(\pi n) - \sin(\pi n)\right], \qquad n \neq 0$$

Since $\sin(\pi n) = 0$ for all n, the contribution of this term to h(n) is 0 except for the n = 0 case, for which the numerator and denominator are both = 0). Therefore, the overall expression for h(n) becomes:

$h(n) = \frac{1}{nT}\left[\cos(\pi n)\right]$ for n ≠ 0 (equation 4.47 in text)

and

$h(n) = 0$ for $n = 0$ (since the integral of equation $(****)$ = 0 for the n = 0 case)

Example: Illustration of Example 4.4 with a Sinusoidal Input

Consider the analog input to the above digitally-implemented bandlimited differentiator to be

$x_c(t) = \cos(\Omega_0 t)$ with $\Omega_0 < \pi/T$ (i.e., the sampling rate is adequate)

The output of the ideal C/D converter will then be:

$x(n) = \cos(\omega_0 n)$ where $\omega_0 = \Omega_0 T < \pi$.

As already shown, the DTFT of $x(n) = \cos(\omega_0 n)$ can be expressed as

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\left[\pi\delta(\omega - \omega_o + 2\pi r) + \pi\delta(\omega + \omega_o + 2\pi r)\right]$$

For $|\omega| < \pi$ (the frequency range that will be involved in determining y(n)), this can be expressed as

$$X(e^{j\omega}) = \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o) \qquad |\omega| < \pi$$

The output of the discrete-time part ("middle part") of the overall system of Figure 4.11 is therefore

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y(e^{j\omega}) = \frac{j\omega}{T}\left[\pi\delta(\omega-\omega_o)\pi + \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{j\omega_o\pi}{T}\left[\delta(\omega-\omega_o)\right] - \frac{j\omega_o\pi}{T}\left[\delta(\omega+\omega_o)\right] \quad \text{for } |\omega| < \pi$$

If y(n) is the input to an ideal D/C converter, the overall analog output is then

$$Y_r(j\Omega) = TY(e^{j\omega})\Big|_{\varpi=\Omega T} = TY(e^{j\Omega T}), \quad |\Omega| < \frac{\pi}{T}$$

$$= T\left[\frac{j\omega_o\pi}{T}\left[\delta(\Omega T-\Omega_o T)\right] - \frac{j\omega_o\pi}{T}\left[\delta(\Omega T+\Omega_o T)\right]\right]$$

$$= T\left[\frac{j\omega_o\pi}{T}\frac{1}{T}\left[\delta(\Omega-\Omega_o)\right] - \frac{j\omega_o\pi}{T}\frac{1}{T}\left[\delta(\Omega+\Omega_o)\right]\right]$$

$$= j\pi\Omega_o\delta(\Omega-\Omega_o) - j\pi\Omega_o\delta(\Omega+\Omega_o)$$

Since the inverse Fourier Transform of $\delta(\Omega-\Omega_o)$ is $\frac{1}{2\pi}e^{j\Omega_o t}$,

the time-domain counterpart of $Y_r(j\Omega)$ is

$$y_r(t) = j\Omega_o\frac{1}{2}e^{j\Omega_o t} - j\Omega_o\frac{1}{2}e^{-j\Omega_o t} = j\Omega_o\frac{(e^{j\Omega_o t} - e^{-j\Omega_o t})}{2}$$

$= -\Omega_o \sin(\Omega_o t)$ which is indeed the derivative of the analog input, $x_c(t)$.

Section 4.5 Continuous-Time Processing of Discrete-Time Systems.

The following system configuration represents continuous-time processing of discrete-time signals:

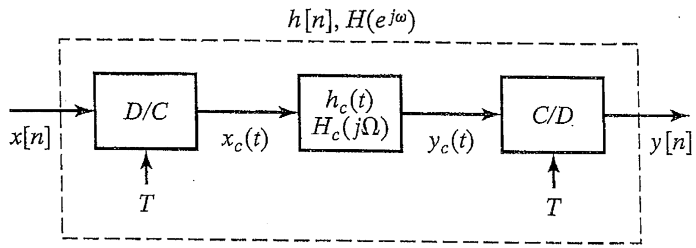


Figure 4.15 Continuous-time processing of discrete-time signals.

(This is complementary to the system considered earlier: Discrete-Time Processing of Continuous-Time Systems)

Note: This approach is not typically used to implement discrete-time systems, but it provides a framework for interpreting certain discrete-time systems.

The ideal D/C converter can be represented by a system that includes an analog impulse generator followed by an ideal low pass filter with cutoff frequency of π/T and a gain of T. (See Fig. 4.7, p. 164). Therefore, $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$. Furthermore, the "reconstruction formula" associated with the Sampling Theorem can be used to relate $x_c(t)$ with x(n), as follows:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

The frequency-domain operations associated with the process represented in Figure 4.15 are as follows:

$$X_c(j\Omega) = TX(e^{j\Omega T}) \qquad |\Omega| < \pi/T$$

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad |\Omega| < \pi/T$$

$$Y(e^{j\omega}) = \frac{1}{T} Y_c(\frac{j\omega}{T}), \qquad |\omega| < \pi$$

Combining the above three relations and using the fact that $\omega = \Omega/T$,

$$Y(e^{j\omega}) = \frac{1}{T}\left[H_c(\frac{j\omega}{T}) X_c(\frac{j\omega}{T}) \right]$$

$$= \frac{1}{T}\left[H_c(\frac{j\omega}{T}) TX(e^{j\frac{\omega}{T}T}) \right]$$

$$= H_c(\frac{j\omega}{T}) X(e^{j\omega}), \qquad |\omega| < \pi$$

Therefore, we can write

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

where

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}) \qquad |\omega| < \pi$$

Now consider an example which makes use of the above system structure:

Example 4.7 Noninteger Delay (essentially an interpolation between samples of a discrete time signal)

Consider the analog part (the middle part) of the system of Figure 4.15 to be

$$H_c(j\Omega) = e^{-j\Omega\Delta T}$$

Then $y_c(t) = x_c(t - \Delta T)$

where Δ is not necessarily an integer.

The overall output y(n) corresponds to sampled values of $y_c(t)$ at t = integer multiples of T. That is,

$$y(n) = y_c(nT) = x_c(nT - \Delta T) \ .$$

In the frequency domain,

$$H(e^{j\omega}) = H_c(j\Omega)\Big|_{\Omega=\frac{\omega}{T}} \qquad |\omega| \le \pi$$

$$= e^{-j\Omega\Delta T}\Big|_{\Omega=\frac{\omega}{T}} \qquad |\omega| \le \pi$$ which can be written as $H(e^{j\omega}) = e^{-j\omega\Delta}$, $|\omega| \le \pi$

Figure 4.16 relates y(n) to $x_c(t)$ for the case of $\Delta = 1/2$:

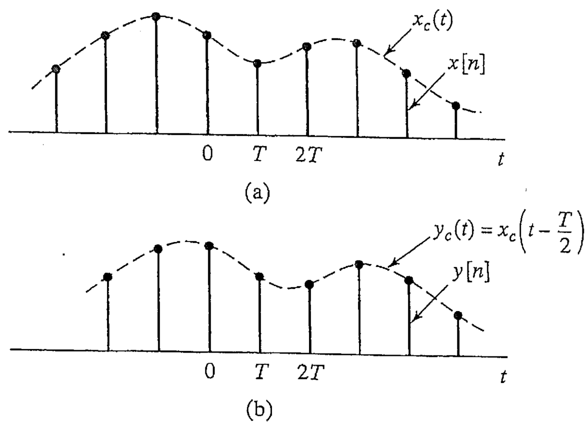


Figure 4.16 Continuous-time processing of the discrete-time sequence in part (a) can produce a new sequence with a "half-sample" delay, as in part (b).

Another time-domain interpretation of the above. Combining

$$y(n) = x_c(nT - \Delta T) = x_c(t - \Delta T)\Big|_{t=nT}$$

and

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t - nT)]}{\pi(t - nT)/T}$$

and changing the summation index from n to k gives:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \frac{\sin[\pi(t - \Delta T - kT)/T]}{\pi(t - \Delta T - kT)/T}\Bigg|_{t=nT}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \frac{\sin[\pi(n-k-\Delta)]}{\pi(n-k-\Delta)}$$ (assuming that Δ is not an integer)

This represents a convolution of x(k) with

$$h(n) = \frac{\sin\left[\pi(n-\Delta)\right]}{\pi(n-\Delta)}$$, all n (where again Δ is not an integer)

Note that the above h(n) represents in IIR filter. If Δ is an integer n_o , then

$$y(n) = x_c(nT - n_oT) = x_c((n-n_o)T)$$

which corresponds to

$$h(n) = \delta(n - n_0)$$

Example 4.8

Consider a discrete-time system whose output is the (M+1)-point moving average of its input. That is,

$$y(n) = \frac{1}{M+1} \sum_{k=0}^{M} x(n-k)$$

The corresponding unit sample response is

$$h(n) = \frac{1}{M+1} \sum_{k=0}^{M} \delta(n-k)$$

The frequency response of this system is

$$H(e^{j\omega}) = \sum_{k} h(k) e^{-j\omega k}$$

$$= \frac{1}{(M+1)} \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2} \qquad |\omega| < \pi$$

This can be represented as the cascade of two systems $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ where

$$H_1(e^{j\omega}) = \frac{1}{(M+1)}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} \quad \text{and} \quad H_2(e^{j\omega}) = e^{-j\omega M/2}$$

The overall system is shown below in Figure 4.17:

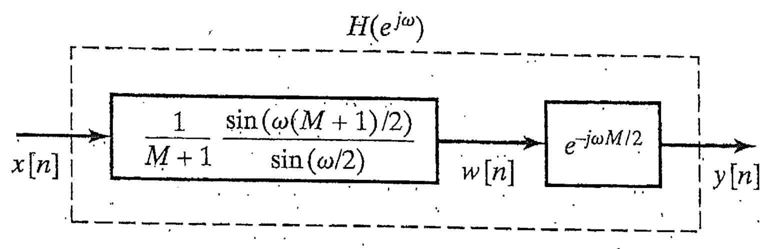
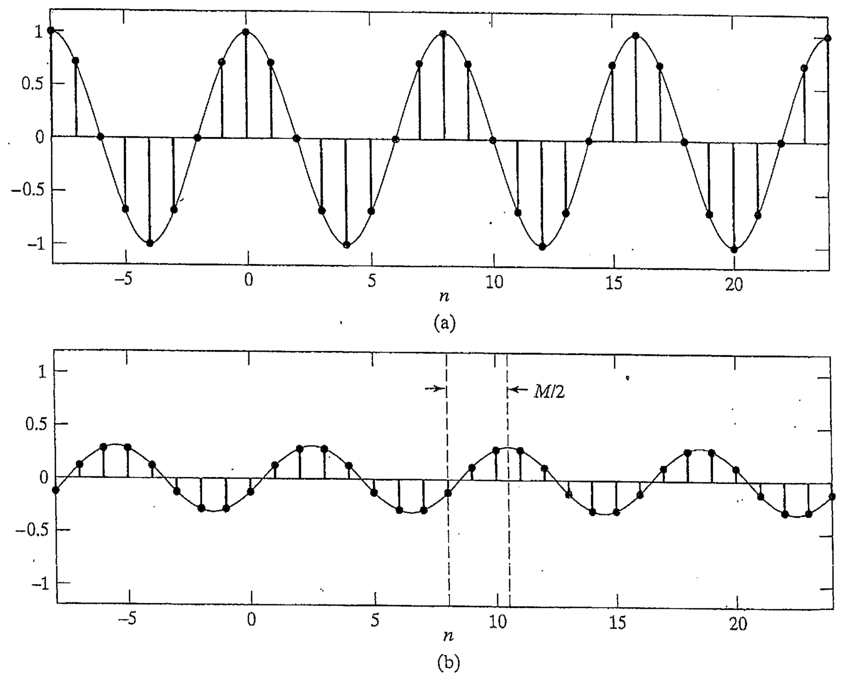


Figure 4.17 The moving-average system represented as a cascade of two systems.

The magnitude response of the filter depends entirely upon $H_1(e^{j\omega})$.
$H_2(e^{j\omega})$ in a linear phase term that represents a pure time delay.

If M is an even integer, then the delay corresponds to an integer no. of samples (M/2 samples).
If M is an odd integer, then the delay of M/2 corresponds to delay of an "integer plus one-half".

This is shown in Figure 4.18 below, for the case of M=5 (a 6-point moving average) when the input is $x(n) = \cos.25\pi n)$.



4.18 Illustration of moving-average filtering. (a)Input signal $x[n] = \cos(0.25\pi n)$.
(b) Corresponding output of six-point moving-average filter.

The time-domain expression for the output can be obtain by first writing the input as

$$\cos(.25\pi n) = \frac{e^{j.25\pi n} + e^{-j.25\pi n}}{2}$$

and then

$$y(n) = H(e^{j.25\pi})\frac{1}{2}e^{j.25\pi n} + H(e^{-j.25\pi})\frac{1}{2}e^{-j.25\pi n}$$

For M = 5, this resolves to

$$y(n) = \frac{1}{2}\frac{\sin[3(.25\pi)]}{6\sin(.125\pi)}e^{-j(.25\pi)5/2}e^{j.25\pi n} + \frac{1}{2}\frac{\sin[3(-.25\pi)]}{6\sin(-.125\pi)}e^{j(.25\pi)5/2}e^{-j.25\pi n}$$

$$= .1540e^{j.25\pi(n-2.5)} + .1540e^{-j.25\pi(n-2.5)}$$

$$= .308\cos[.25\pi(n-2.5)]$$ (Note the delay by a non-integer no. of samples.)