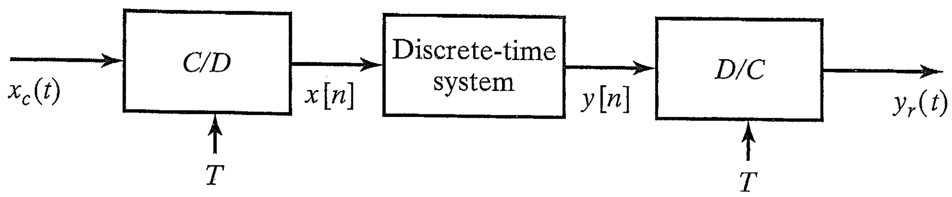
ECE 8440 - Unit 4

Digital Processing of Analog Signals--Non-Ideal Case (See section 4.8)

Before considering the non-ideal case, recall the ideal case:

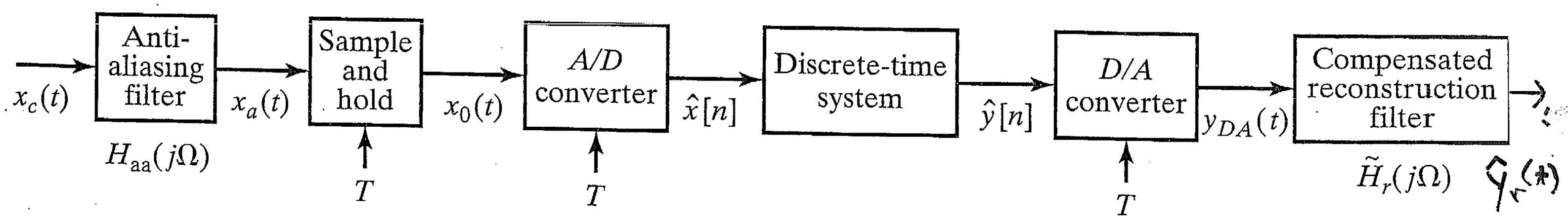


(a)

Assumptions involved in ideal case:

- no aliasing
- no quantization errors in converting from continuous time to discrete time
- ideal impulse generator and ideal low-pass filters used in implementing D/C converter

- Realizable system used to approximate the ideal system is shown below:



(b)

Figure 4.47 (b) Digital Processing of Analog Signals

<u>Use of anti-aliasing filter</u>:

Recall figure from ECE 667 text (Ludeman):

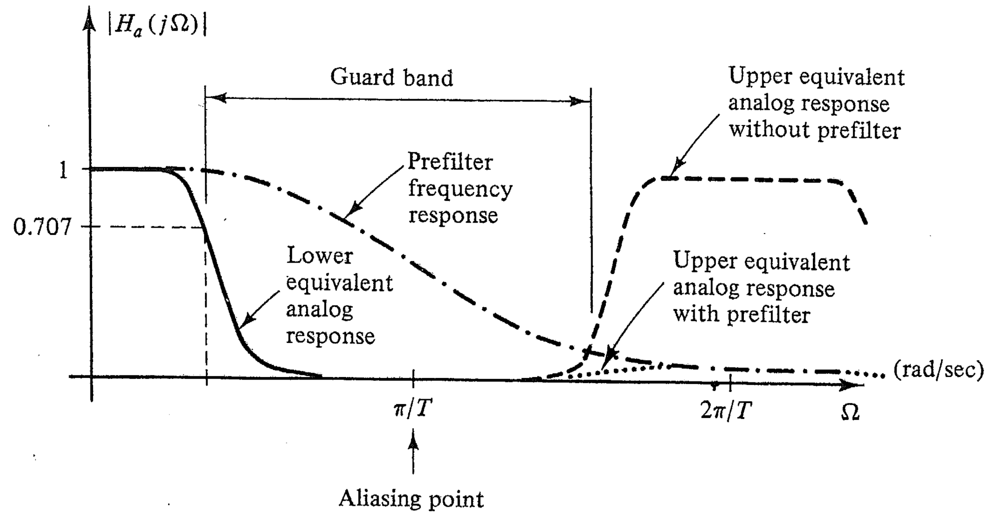


Figure 1.29 Guard band and prefilter frequency response.

Ideally, the anti-aliasing filter should satisfy

$$H_{aa}(j\Omega) = 1 \quad \text{for} \quad |\Omega| < \Omega_c \le \frac{\pi}{T} \qquad \text{(equation 4.118)}$$

$$H_{aa}(j\Omega) = 0 \quad \text{for} \quad |\Omega| > \Omega_c$$

If anti-aliasing filter is ideal (and the D/C converter is ideal), then the <u>overall</u> frequency response between the input $\mathbf{x_c(t)}$ and the output $\hat{y}_r(t)$ is

$$H_{eff}(j\Omega) = H(e^{j\Omega T}) \qquad \text{for } |\Omega| < \Omega_c \le \frac{\pi}{T}$$

$$H_{eff}(j\Omega) = 0 \qquad \text{for } |\Omega| > \Omega_c$$

However, since the anti-aliasing filter is not perfectly flat the actual overall frequency response of the above system is

$$H_{eff}(j\Omega) \simeq H_{aa}(j\Omega)H(e^{j\Omega T})$$

Note: The closer Ω_c is to **π/T** the more difficult it is to approximate an ideal anti-aliasing filter over the desired filter's passband.

Problems with using "approximately ideal" anti-aliasing filters :

- expensive
- generally have highly non-linear phase.

Alternative approach:

Step 1. "over-sample" the input so that so that the frequencies of interest in the signal, after sampling, satisfy

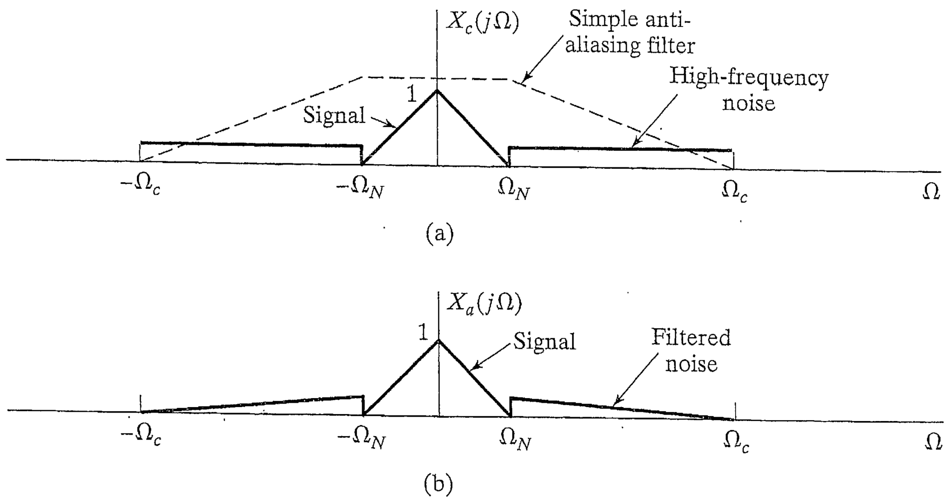
$$\omega < \omega_N << \pi$$

Stated in terms of the analog input, select the sampling rate 1/T to satisfy

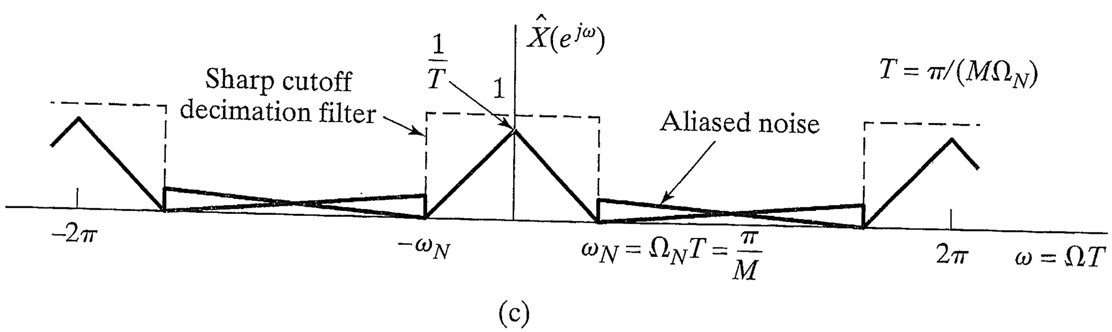
$$\left(\frac{2\pi}{T} - \Omega_c\right) > \Omega_N$$

where Ω_N is the highest frequency of interest in $x_c(t)$ and Ω_c is the highest frequency that the "cheap" ("simple”) anti-aliasing filter passes. (See Figure 4.50 on next slide.) Assume that this involves a sampling rate that is M times the Nyquist rate. That is,

$$\frac{1}{T} > M\left(\frac{\Omega_N}{\pi}\right)$$



Step 2. Apply a sharp cut-off digital filter to remove high frequencies out of the range of interest. (The digital filter can also have linear phase.)



Step 3. Apply down-sampling by the value of M that is used in Step 1.

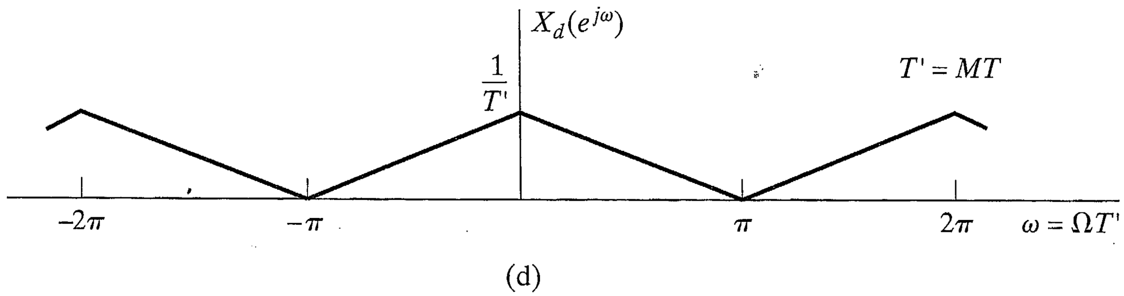


Figure 4.50 Use of oversampling followed by decimation in C/D conversison.

Analog-to-Digital Conversion (practical realization of C/D conversion)

Non-ideal properties:

-conversion cannot be performed instantaneously

-therefore, a Sample/Hold circuit is typically used

-quantization error arise due to using a finite no. of bits to represent signals that have a continuous range of possible values

If an A/D converter uses B output bits to represent a signal whose possible range of input values is from 0 to X_m, the "step size" (resolution) of the converter is

$$\frac{X_m}{2^B} = \Delta$$

Similarly, if the A/D converter uses B+1 bits to represent <u>inputs which can be positive or negative</u> with magnitude up to X_m , then the step size is

$$\frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B} = \Delta$$

As can be seen in the figure below, the maximum quantization error for an A/D converter with step size Δ is $\Delta/2$ (as long as the output is not forced into saturation in the top or bottom levels).

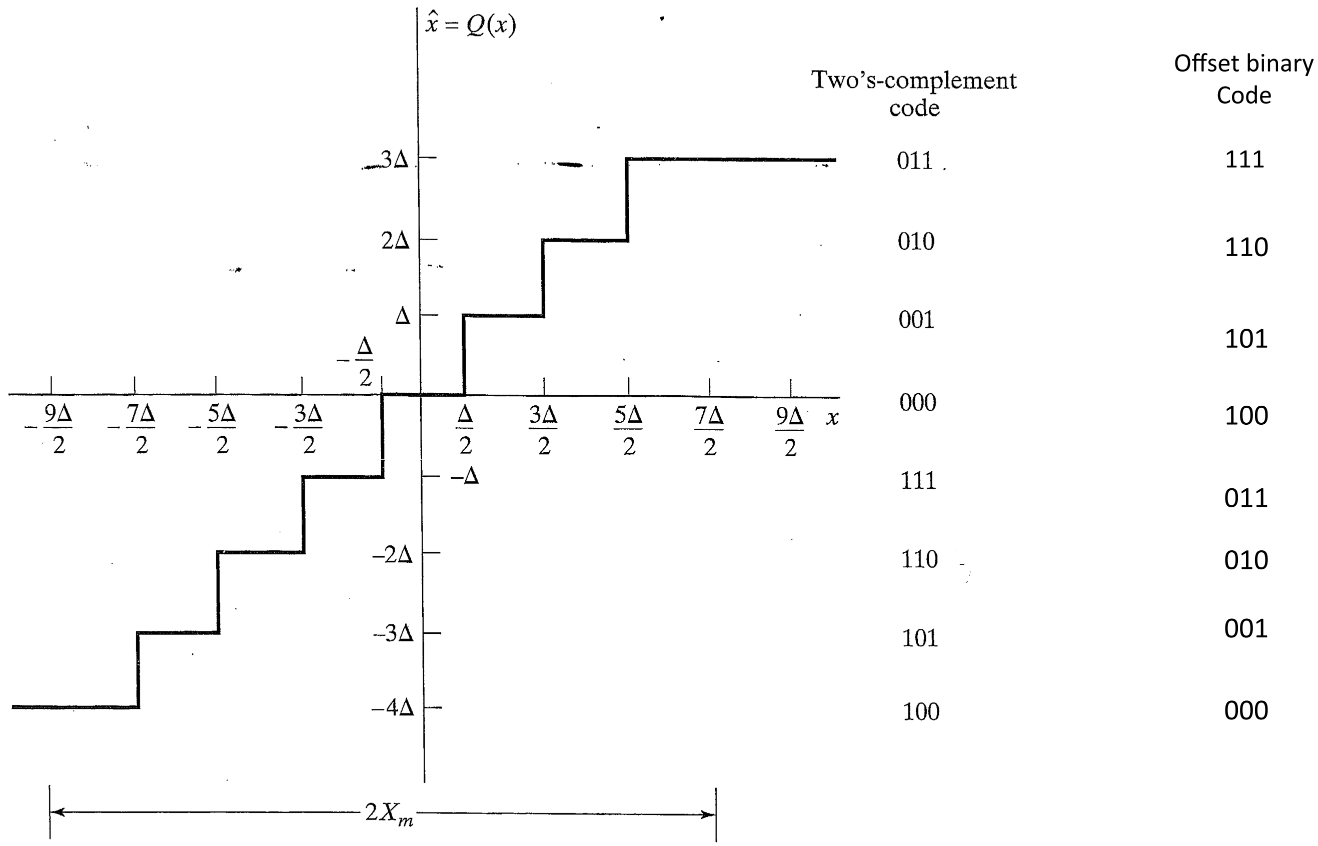


Figure 4.54 Typical quantizer for A/D conversion

Analysis of Quantization Errors due to A/D Conversion

If the perfectly represented discrete time signal is x(n) and the actual output of the A/D converter is $\hat{x}(n)$, then the A/D quantization error e(n) is defined as

$$e(n) = \hat{x}(n) - x(n).$$

Feeding the output of an A/D converter into the digital filter effectively introduces a noise component (due to quantization) along with the desired input signal

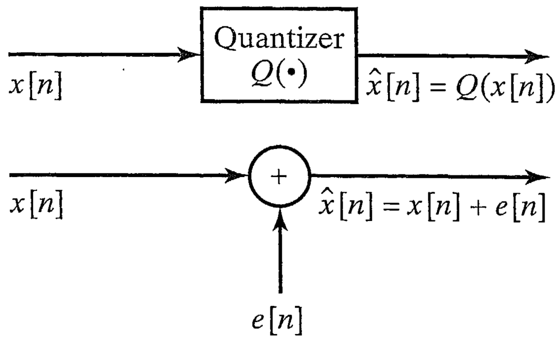


Figure 4.56 Additive noise model for quantizer.

<u>Size of quantization error:</u>

As seen on the previous page, the quantization error range is:

$$-\Delta/2 < e(n) < \Delta/2$$

This range of error values is valid as long as the input signal stays within the allowed range. For a "mid-tread" A/D converter as shown on the previous slide, this condition is

$$\left(-X_m - \Delta/2\right) < x(n) < \left(X_m - \Delta/2\right)$$ (Refer to fig. 4.54)

If x(n) exceeds this range, then larger, "clipping" quantization errors occur.

<u>In order to perform a statistical evaluation of the effect of A/D quantization errors, several assumptions are typically made</u>:

1. e(n) is the realization of stationary random process
2. e(n) is uncorrelated with the input x(n)
3. e(n) is uncorrelated with e(m) for n ≠ m. That is, e(n) is a "white noise" process
4. e(n) has a uniform probability density function

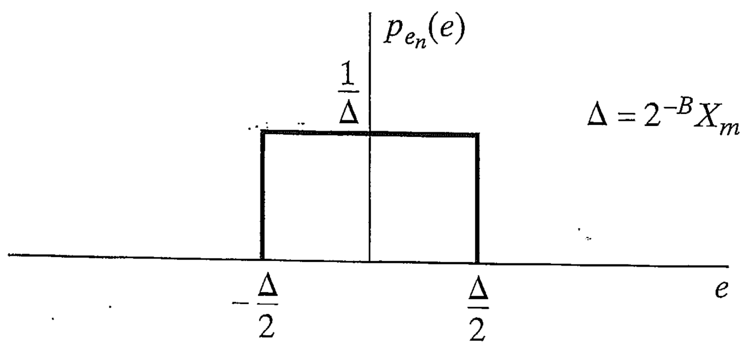


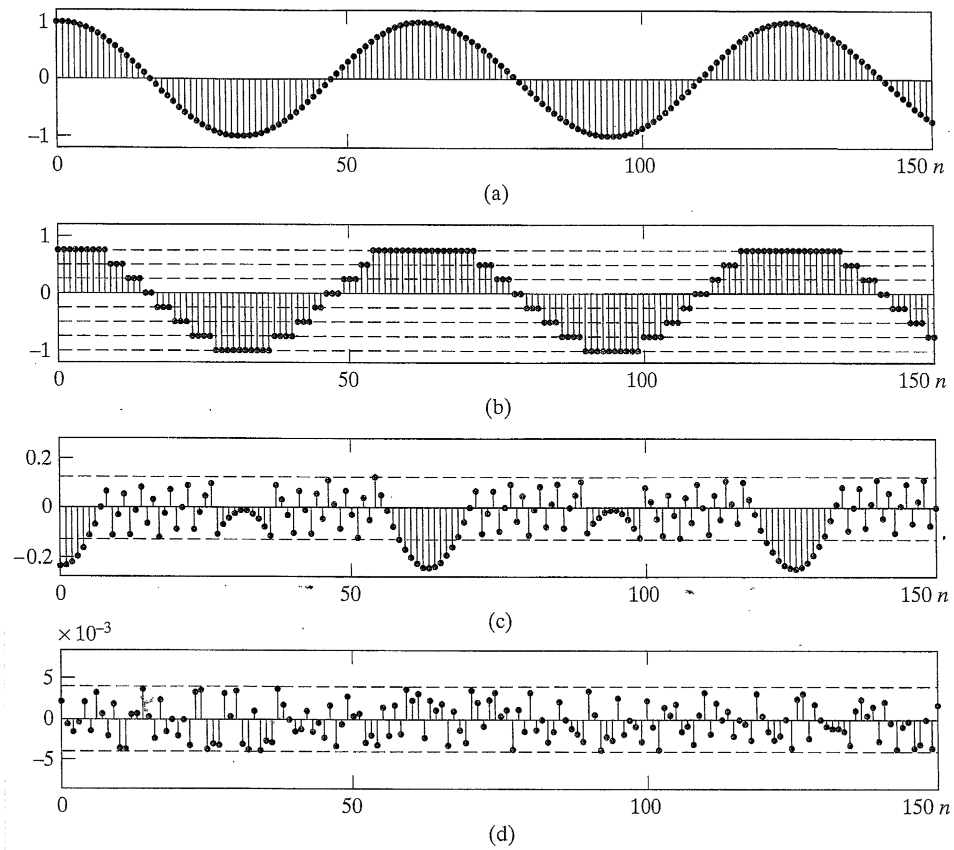
Figure 4.58 Probability density Function of quantization error for a Rounding quantier such as that of Figure 4.54.

These assumptions are reasonable when:

- the input signal is "sufficiently complex" (e.g., speech, music)
  (example of "not sufficiently complex": a step function or a low order polynomial, such as a line)
- the input amplitude typically traverses many quantization steps from sample to sample.

The second condition (transversal of many quantization steps) becomes easier to satisfy as the number of quantization levels is increased, as seen in the figure below.

Note the "clipping'' error in part c of the figure.



(a) Unquantized samples of the signal x(n)=.99kcos(π/10)

$\Delta = \frac{2}{2^3} = \frac{1}{4}$

(b) Quantized samples of signal in part (a) with a 3-bit quantizer

(c) Quantization error sequence for 3-bit quantization of signal of part (a)

(d) Quantization error sequence for 8-bit quantization of signal of part (a)

Figure 4.57 Example of quantization noise

Statistical representation of quantization errors:

If the above assumption that e(n) is uniformly distributed is valid, then the mean value of e(n) is 0, and the variance of e(n) can be found as follows:

$$\sigma_e^2 = \int_{-\Delta/2}^{\Delta/2} e^2\left(\frac{1}{\Delta}\right)de = \frac{\Delta^2}{12}$$

Since $\Delta = \frac{X_m}{2^B}$,

$$\sigma_e^2 = \frac{1}{12}\left(\frac{X_m}{2^B}\right)^2 = \frac{X_m^2 2^{-2B}}{12}$$

Often the effect of quantization error is expressed in terms of db of the signal-to-noise ratio:

$$\begin{aligned} SNR &= 10\log_{10}\frac{\sigma_x^2}{\sigma_e^2} \\ &= 10\log_{10}\frac{\sigma_x^2}{X_m^2 2^{-2B}/12} \\ &= 10\log_{10}\left(\frac{12 \cdot 2^{2B}\sigma_x^2}{X_m^2}\right) \end{aligned}$$

In other to have an input signal to the A/D whose typical input values span the entire input range of the A/D, but for which there little chance of clipping, we would typically adjust the input level to satisfy:

$$X_m = 4\sigma_x$$

Then the resulting SNR becomes

$$SNR = 10\log_{10}\left(\frac{12\cdot 2^{2B}\left(\frac{X_m}{4}\right)^2}{X_m^2}\right)$$

$$= 10\log_{10}[3\cdot 2^{2B-2}]$$
$$= 10[\log_{10}(3) + (2B-2)\log_{10}(2)]$$
$$= 10[.477 + (2B-2)(.301)]$$
$$= 6.02B - 1.25db$$
$$\approx 6B - 1.25db$$

Therefore, each additional bit in an A/D output contributes approximately 6 db to the SNR performance.

Example:

If 90 - 96 db of SNR is required (as in high quality music recording), then the required no. of bits in the A/D output can be calculated as

$$B = \frac{90+1.25}{6} = 15.2$$ (for 90 db SNR)

or $$B = \frac{96+1.25}{6} = 16.2$$ (for 96 db SNR)

D/A Conversion

Recall that the ideal D/C conversion process can be represented in the time domain as

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n)\frac{\sin[\frac{\pi}{T}(t-nT)]}{\frac{\pi}{T}(t-nT)} \qquad \text{(equation 4.140)}$$

(This is the "reconstruction formula" which is a by-product of developing the Sampling Theorem.)

Mathematically, this is equivalent to generating a train of analog impulse scaled by x(n), then feeding the pulse train into an ideal low-pass filter, as shown below and derived on the next slide.

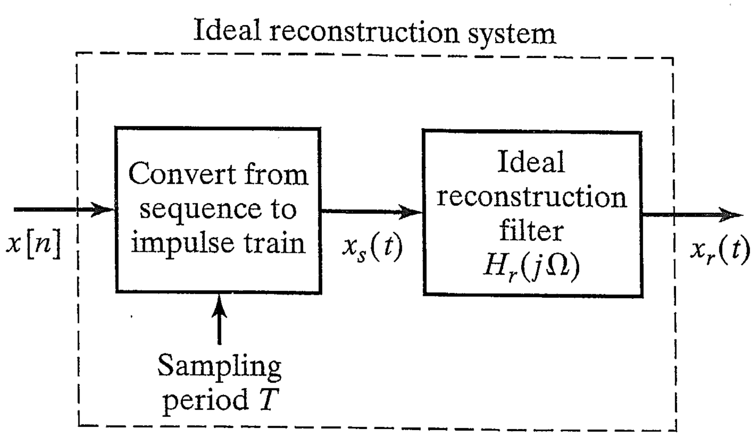
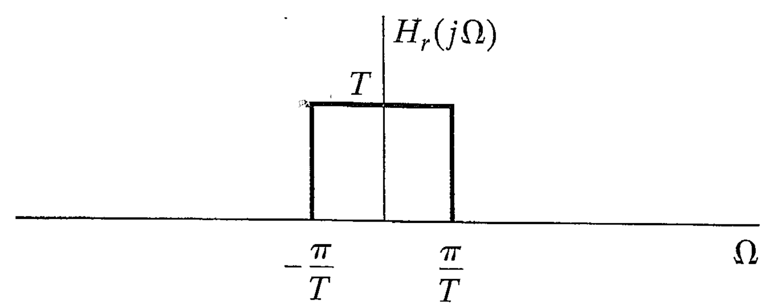


Figure 4.7(a) Block diagram of an deal bandlimited signal reconstruction system.



(b) Frequency response of an ideal reconstruction filter.

This pulse train can be represented as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT). \qquad \text{(equation 4.22)}$$

The impulse response of an ideal low-pass filter having cutoff equal to π/T and with gain of T is

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}. \qquad \text{(equation 4.24)}$$

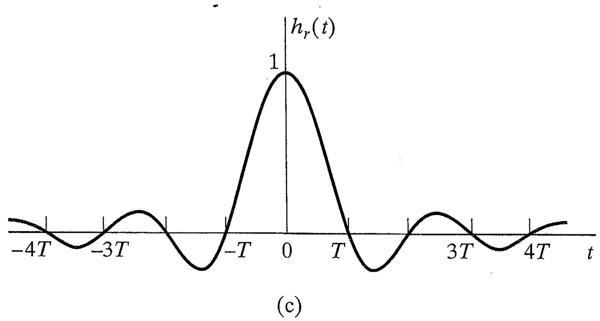


Figure 4.7 (c) Impulse response of an Ideal reconstruction filter.

The response of this low-pass filter to the input $x_s(t)$ is

$$x_r(t) = \int_{-\infty}^{\infty} x_s(\tau) h_r(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n)\delta(\tau - nT) h_r(t - \tau) d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(\tau - nT) h_r(t - \tau)\, d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(n) h_r(t - nT) \qquad \text{(equivalent to equation 4.25)}$$

To obtain a frequency domain description of the ideal reconstruction process, take the continuous time Fourier Transform of of the output $x_r(t)$:

$$X_r(j\Omega) = \int_{-\infty}^{\infty} x_r(t)e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x(n)h_r(t-nT)\right)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x(n)\int_{-\infty}^{\infty} h_r(t-nT)e^{-j\Omega t}dt$$

If we let $\tau = t - nT$, the above can be written as

$$= \sum_{n=-\infty}^{\infty} x(n)\int_{-\infty}^{\infty} h_r(\tau)e^{-j\Omega(\tau+nT)}d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega nT}\int_{-\infty}^{\infty} h_r(\tau)e^{-j\Omega\tau}d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega nT}H_r(j\Omega)$$

$$= X(e^{j\omega})H_r(j\Omega)$$

So $X_r(j\Omega) = X(e^{j\omega})H_r(j\Omega)$

where $H_r(j\Omega)$ is an ideal low-pass filter with gain = T, as was shown previously in figure 4.7.

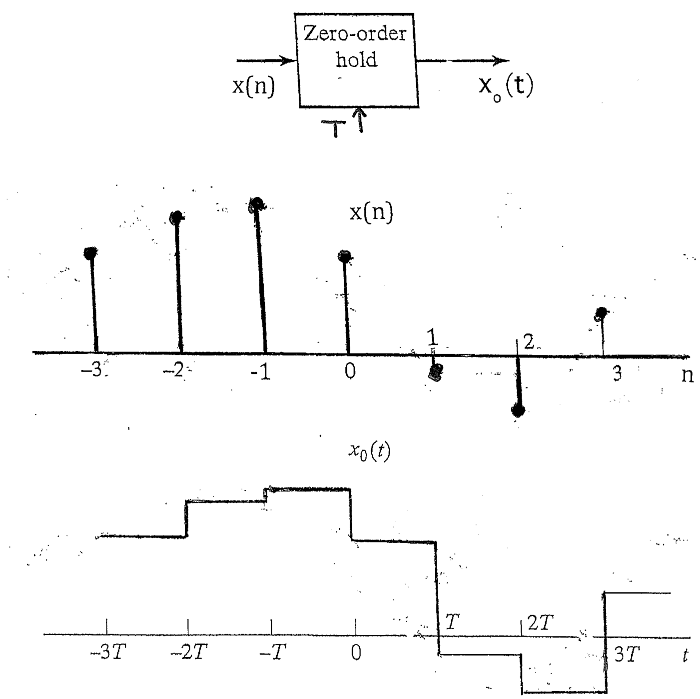
Since $\omega = \Omega T$, the above equation can be written as

$$X_r(j\Omega) = X(e^{j\Omega T})H_r(j\Omega) \qquad \text{(equation 4.28)}$$

D/A converters

A D/A converter is a practical realization (and approximation) of the ideal D/C converter.

A typical D/A converter uses a zero-order hold to preserve the most recent analog conversion value, as shown in the figure below.



The operation of a zero-order hold can be represented mathematically as

$$x_o(t) = \sum_{n=-\infty}^{\infty} x(n)h_o(t-nT)$$

where

$$h_o(t) = 1, \quad 0 < t < T$$

and

$$h_o(t) = 0, \quad \text{otherwise}$$

It is useful to note that the generation of $x_o(t)$ from x(n) can also be represented by a convolution of a weighted impulse train with $h_o(t)$:

$$x_o(t) = h_o(t) * \sum_{n=-\infty}^{\infty} x(n)\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(n)h_o(t-nT).$$

To view the effect of the zero-order hold in the frequency domain, take the Fourier Transform of $h_o(t)$:

$$H_o(j\Omega) = \int_{-\infty}^{\infty} h_o(t)e^{-j\Omega t}\,dt = \int_0^T 1\, e^{-j\Omega t}\,dt$$

$$= \left.\frac{e^{-j\Omega t}}{-j\Omega}\right|_0^T = -\frac{\left[e^{-j\Omega T}-1\right]}{j\Omega} = \frac{\left[1-e^{-j\Omega T}\right]}{j\Omega}$$

$$= e^{-\frac{j\Omega T}{2}}\frac{\left[e^{\frac{j\Omega T}{2}}-e^{-\frac{j\Omega T}{2}}\right]}{j\Omega} = \frac{2}{\Omega}\sin\left(\frac{\Omega T}{2}\right)e^{-\frac{j\Omega T}{2}} \qquad \text{(equation 4.150)}$$

To compensate for the frequency response contributions by the zero-order hold, we can append the following <u>compensated reconstruction filter</u> to the output of a D/A converter which utilizes zero-order hold.

$$\tilde{H}_r(j\Omega) = \frac{T}{H_o(j\Omega)} = \frac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2} \qquad \text{for } \Omega < \frac{\pi}{T} \qquad \text{(equation 4.151)}$$

$$= 0 \qquad \text{otherwise}$$

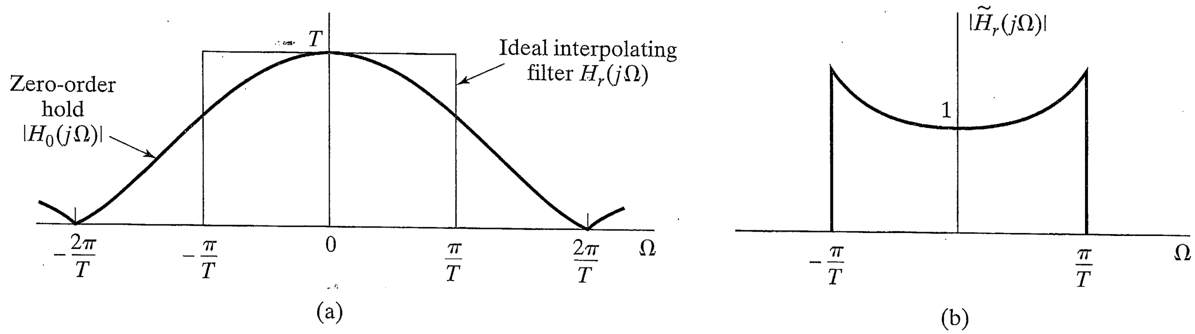
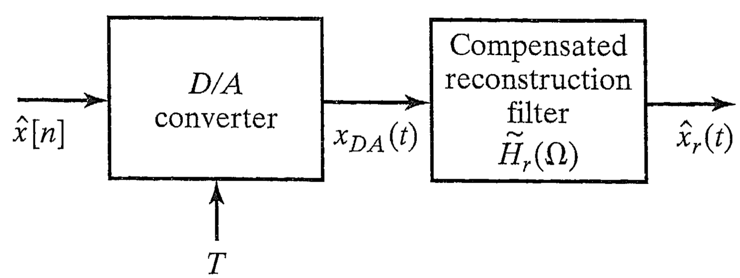


Figure 4.63 (a) Frequency Response of zero-order-hold with ideal interpolating filter.

(b) Ideal compensated reconstructed filter for use with zero-order-hold output.

The "compensated reconstruction filter" $\tilde{H}_r(j\Omega)$ is cascaded with the A/D converter to compensate for the frequency shaping inherent in the zero-order hold, as shown in the figure below.



(c) Figure 4.64 Physical configuration for D/A conversion.

For a signal processing system consisting the following units:

- anti-aliasing filter, $H_{aa}(j\Omega)$
- A/D conversion
- linear filtering, $H(e^{j\omega}) = H(e^{j\Omega T})$
- D/A conversion using zero-order hold, $H_o(j\Omega)$
- compensated reconstruction filter that compensates for zero-order hold, $\tilde{H}_r(j\Omega)$

The overall frequency response between the continuous-time and the continuous-time output is

$$H_{eff}(j\Omega) = \tilde{H}_r(j\Omega)H_o(j\Omega)H(e^{j\Omega T})H_{aa}(j\Omega)$$

Recall: The output of the A/D converter includes A/D quantization noise and can be represented as

$$\hat{x}(n) = x(n) + e(n)$$

where x(n) is the unquantized signal value and e(n) is the A/D quantization error.

To be shown later: If the A/D quantization error is a white noise signal with variance

$$\sigma_e^2(n) = \frac{\Delta^2}{12}$$

then the power spectrum of the output noise due to A/D quantization is

$$P_e(j\Omega) = |\tilde{H}_r(j\Omega)H_o(j\Omega)H(e^{j\Omega T})|^2 \sigma_e^2$$