ECE 8440 - Unit 5

Definition of a random signal and a discrete-time random process (See Section 2.10 and Appendix A.1-A.4)

• Each individual sample of a random signal x(n), i.e., the value of x(n) for some n, is assumed to be an outcome of some underlying random variable $\mathbf{x_n}$.

• The entire signal is represented by a set of such random variables (one for each time index,) and is called a random process.

• Formally, we say that a random process is an indexed family of random variables $\{\mathbf{x_n}\}$ characterized by the set of individual and joint probability distributions of all the random variables the random process consists of.

• A particular sequence of signal values x(n) is often considered to be one of an ensemble of sample sequences associated with an underlying random process.

• An individual random variable $\mathbf{x_n}$ is completely characterized by its probability distribution function

$P_{x_n}(x_n,n) =$ Probability $[\mathbf{x_n} \le x_n]$. (equation A.1)

If $\mathbf{x_n}$ takes on a continuous range of values, then it can be specified by its probability density function

$$p_{x_n}(x_n,n) = \frac{\partial}{\partial x_n}\left(P_{x_n}(x_n,n)\right).$$

The interdependence of two random variables $\mathbf{x_n}$ and $\mathbf{x_m}$ of a random process can be described by the joint probability distribution function

$$P_{x_n,x_m}(x_n,n,x_m,m) = \text{Probability } [\mathbf{x_n} \le x_n \text{ and } \mathbf{x_m} \le x_m]$$

and by the joint probability density function

$$p_{x_n,x_m}(x_n,n,x_m,m) = \frac{\partial^2}{\partial x_n \partial x_m}\left(P_{x_n,x_m}(x_n,n,x_m,m)\right).$$

Statistical Independence

Two random variables $\mathbf{x_n}$ and $\mathbf{x_m}$ are statistically independent if knowledge of the value of one does not affect the probability density of the other. If $\mathbf{x_n}$ and $\mathbf{x_m}$ are statistically independent, then the joint probability distribution function of $\mathbf{x_n}$ and $\mathbf{x_m}$ can be expressed as:

$$P_{x_n,x_m}(x_n,n,x_m,m) = P_{x_n}(x_n,n) \cdot P_{x_m}(x_m,m) \qquad m \neq n \qquad \text{(equation A.6)}$$

Likewise,

$$p_{x_n,x_m}(x_n,n,x_m,m) = p_{x_n}(x_n,n) \cdot p_{x_m}(x_m,m)$$

(The above properties extend to an arbitrary number of statistically independent random variables.)

Stationary Random Processes

If all the individual and joint probability distributions for the random variables that make up a random process are independent of a shift of the time origin, the random process is said to be stationary. For example, if x_n and x_m are components of a stationary random process, then their joint probability distribution must satisfy

$$P_{x_{n+k},x_{m+k}}(x_{n+k},n+k,x_{m+k},m+k) = P_{x_n,x_m}(x_n,n,x_m,m) \quad \text{for all k} \qquad \text{(equation A.7)}$$

Correspondingly, the following must be satisfied:

$$p_{x_{n+k},x_{m+k}}(x_{n+k},n+k,x_{m+k},m+k) = p_{x_n,x_m}(x_n,n,x_m,m)$$

Important Considerations

- It is not generally useful to apply the DTFT directly to a random signal.
- However, it is often useful to apply the DTFT to averages such as the autocorrelation sequence or the autocovariance sequence associated with a random signal.

Important Averages

The average (or mean or expected value) of a random variable x_n is defined as

(equation A.8)

$$m_{x_n} = E\{\mathbf{x}_n\} = \int_{-\infty}^{\infty} x \cdot p_{x_n}(x,n)\, dx$$

where $E\{\mathbf{x}_n\}$ denotes the mathematical expectation operator. In general, the mean may be dependent on n.

If $g(x_n)$ is a <u>single valued function of a random variable</u> $\mathbf{x_n}$ then $g(x_n)$ is also a random variable. The set of random variables $\{g(x_n)\}$, $-\infty < n < \infty$, is a new random process.

The average of the new random variable $g(x_n)$ is given by

$$E\{g(\mathbf{x}_n)\} = \int_{-\infty}^{\infty} g(x)\, p_{x_n}(x,n)\, dx \qquad \text{(equation A.9)}$$

If the random variable $g(x_n)$ is <u>discrete</u> (if $g(x_n)$ has quantized values), then

$$E\{g(\mathbf{x}_n)\} = \sum_x g(x)\, \hat{p}_{x_n}(x,n) \qquad \text{(equation A.10)}$$

The expected value of a <u>function</u> of two random variables $\mathbf{x_n}$ and $\mathbf{y_n}$ is defined as

$$E(g(\mathbf{x}_n, \mathbf{y}_m)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, p_{x_n,y_m}(x,n,y,m)\, dx\, dy \qquad \text{(equation A.11)}$$

<u>Uncorrelated Random Variables</u>

Two random variables $\mathbf{x_n}$ and $\mathbf{y_m}$ are <u>uncorrelated (linearly independent)</u> if

$$E(\mathbf{x}_n \mathbf{y}_m) = E(\mathbf{x}_n)E(\mathbf{y}_m) \qquad \text{(equation A.12)}$$

- Statistically independent random variables are also uncorrelated.
- The converse of the above statement is not true: uncorrelated random variables are not necessarily statistically independent.

The variance of a random variable $\mathbf{x}_n$ is defined as

$$\text{var}[\mathbf{x}_n] = \sigma^2_{x_n} = E\left\{ |(\mathbf{x}_n - m_{x_n})|^2 \right\} \quad \text{(equation A.15)}$$

If the random variable $\mathbf{x}_n$ is real-valued, this becomes

$$\text{var}[\mathbf{x}_n] = \sigma^2_{x_n} = E\left\{ (\mathbf{x}_n - m_{x_n})^2 \right\}$$

Note that the right side of above expression can be written as

$$E\left\{ (\mathbf{x}_n - m_{x_n})^2 \right\} = E\left\{ (\mathbf{x}_n^2 - 2\mathbf{x}_n m_{x_n} + m^2_{x_n}) \right\}$$

$$= E(\mathbf{x}_n^2) - 2m_{x_n} E(\mathbf{x}_n) + m^2_{x_n}$$

$$= E(\mathbf{x}_n^2) - m^2_{x_n}$$

Note: For the more general case where $\mathbf{x}_n$ is complex-valued, the expression for the variance becomes

$$\text{var}[\mathbf{x}_n] = \sigma^2_{x_n} = E(|\,\mathbf{x}_n\,|^2) - |\,m_{x_n}\,|^2 \quad \text{(equation A.16)}$$

The autocorrelation sequence of the random process $\{\mathbf{x}_n\}$ is defined as

$$\phi_{xx}(n,m) = E\left\{ \mathbf{x}_n \mathbf{x}_m^* \right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_n x_m^* p_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m)\, dx_n\, dx_m \quad \text{(equation A.17)}$$

The autocovariance sequence of the random process $\{\mathbf{x}_n\}$ is defined as

$$\gamma_{xx}(n,m) = E\left\{ (\mathbf{x}_n - m_{x_n})(\mathbf{x}_m - m_{x_m})^* \right\} \quad \text{(equation A.18)}$$

$$= E\left\{ \mathbf{x}_n \mathbf{x}_m^* - m_{x_n} \mathbf{x}_m^* - m_{x_m}^* \mathbf{x}_n + m_{x_n} m_{x_m}^* \right\}$$

$$= E(\mathbf{x}_n \mathbf{x}_m^*) - m_{x_n} E(\mathbf{x}_m^*) - m_{x_m}^* E(\mathbf{x}_n) + m_{x_n} m_{x_m}^*$$

$$= E(\mathbf{x}_n \mathbf{x}_m^*) - m_{x_n} m_{x_m}^*$$

$$= \phi_{xx}(n,m) - m_{x_n} m_{x_m}^*$$

The <u>cross-correlation sequence</u> associated with the random processes $\{\mathbf{x_n}\}$ and $\{\mathbf{y_m}\}$ is defined as

$$\phi_{xy}(n,m) = E(\mathbf{x}_n, \mathbf{y}_m^*)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_n y_m^* p_{x_n,y_m}(x_n,n,y_m,m)\,dx_n\,dy_m \qquad \text{(equation A.20)}$$

The <u>cross- covariance sequence</u> is defined as

$$\gamma_{xy}(n,m) = E\left\{(\mathbf{x}_n - m_{x_n})(\mathbf{y}_m - m_{y_m})^*\right\}$$

$$= E\left\{\mathbf{x}_n\mathbf{y}_m^* - m_{x_n}\mathbf{y}_m^* - m_{y_m}^*\mathbf{x}_n + m_{x_n}m_{y_m}^*\right\}$$

$$= E(\mathbf{x}_n\mathbf{y}_m^*) - m_{x_n}E(\mathbf{y}_m^*) - m_{y_m}^*E(\mathbf{x}_n) + m_{x_n}m_{y_m}^*$$

$$= \phi_{xy}(n,m) - m_{x_n}m_{y_m}^* \qquad \text{(equation A.21)}$$

<u>Wide-Sense Stationary Random Processes</u>

Random variables which are not stationary, but which do satisfy some less restrictive conditions, are said to be <u>wide-sense stationary</u>:

<u>In order to be wide-sense stationary the mean and variance of the random process must be independent of time</u>, and <u>the autocorrelation sequence must be dependent only the time difference between the random variables involved</u>. This is formally stated as:

$$m_x = E(\mathbf{x}_n) \qquad \text{(independent of n)} \qquad \text{(equation A.22)}$$

$$\sigma_x^2 = E\left\{(\mathbf{x}_n - m_x)^2\right\} \qquad \text{(independent of n)} \qquad \text{(equation A.23)}$$

$$\phi_{xx}(n+m,n) = \phi_{xx}(m) = E(\mathbf{x}_{n+m}\mathbf{x}_n^*) \qquad \text{(dependent on m, but not n)} \qquad \text{(equation A.24)}$$

Time averages

The time average of a random process is defined as

$$\langle x_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x_n \qquad \text{(equation A.25)}$$

The time autocorrelation sequence is defined as

$$\langle x_{n+m}, x_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x_{n+m} x_n^* \qquad \text{(equation A.26)}$$

Since $\langle x_n \rangle$ and $\langle x_{n+m}, x_n^* \rangle$ are defined as functions of random variables, they are random variables themselves. Note that if the random process is at least wide-sense stationary, then

$E\{\langle x_n \rangle\} = E\{x_n\} = m_x$ and $E\{\langle x_{n+m}, x_n^* \rangle\} = E\{x_{n+m}, x_n^*\} = \phi_{xx}(m)$

If the random process x_n is also ergodic, then it is also true that

$Var\{\langle x_n \rangle\} = 0$ and $Var\{\langle x_{n+m}, x_n^* \rangle\} = 0.$

Therefore, for this case,

$\langle x_n \rangle = E\{x_n\} = m_x$

A variance of 0 means that the time average associated with almost all sample sequences are equal to the same constant, which is the time-independent average, m_x. Therefore, we can express m_x as the time average for a single sample sequence. That is,

$$\langle x_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n] = E\{x_n\} = m_x \qquad \text{(equation A.27)}$$

Likewise, assuming the same properties (wide sense stationary and ergodic), we can express $\phi_{xx}(m)$ as the following time average associated with a single sample sequence:

$$\langle x_{n+m}, x_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n+m]x^*[n] = E\{x_{n+m}, x_n^*\} = \phi_{xx}(m) \qquad \text{(equation A.28)}$$

In general, a random process for which time averages are equal to ensemble averages is called an ergodic process.

In practice, we typically assume that a given sequence is a sample sequence of a stationary and ergodic random process so that averages can be computed from a single sample sequence. For example, we typically estimate the mean, variance, and autocorrelation using:

$$\hat{m}_x = \frac{1}{L}\sum_{n=0}^{L-1} x(n) \qquad \text{(called the sample mean)} \qquad \text{(equation A.29)}$$

$$\hat{\sigma}_x^2 = \frac{1}{L}\sum_{n=0}^{L-1} | x(n) - \hat{m}_x |^2 \qquad \text{(called the sample variance)} \qquad \text{(equation A.30)}$$

$$\langle x[n+m]x^*[n] \rangle_L = \frac{1}{L}\sum_{n=0}^{L-1} x(n+m)x^*(n) \qquad \text{(“sample autocorrelation”)} \qquad \text{(equation A.31)}$$

Response of LTI Systems to Random Inputs (see section 2.10)

Consider a LTI system with a input x(n) which is a real-valued sequence which is a sample sequence of a wide-sense stationary discrete-time random process. The output y(n) can be also be considered the sample sequence of a random process and is related to the input by

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

The mean of the output process is

$$m_y(n) = E\{y(n)\} = \sum_{k=-\infty}^{\infty} h(k)E\{x(n-k)\}$$

Since we assumed that the input is wide-sense stationary,

$E\{x(n-k)\} = m_x$ (independent of n and k)

Therefore,

$$m_y(n) = m_x \sum_{k=-\infty}^{\infty} h(k) \quad \text{(equation 2.184)}$$

Since the right-hand side is independent of n, so is the left side, and we can write the above equation as:

$$m_y = m_x \sum_{k=-\infty}^{\infty} h(k)$$

Since the above summation is the same as the summation used in the DTFT of h(n) for the case of $\omega = 0$, we can also re-write the above as

$$m_y = m_x H(e^{j0}) \quad \text{(equation 2.185)}$$

The autocorrelation of the output process is

$$\phi_{yy}(n, n+m) = E\left\{y(n)y(n+m)\right\}$$

$$= E\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h(k)x(n-k)h(r)x(n+m-r)\right\}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\sum_{r=-\infty}^{\infty} h(r)E\left\{x(n-k)x(n+m-r)\right\}$$

Since the input was assumed to be wide-sense stationary,

$$E\left\{x(n-k)x(n+m-r)\right\} = \phi_{xx}(m+k-r)$$

Therefore,

$$\phi_{yy}(n,n+m) = \sum_{k=-\infty}^{\infty} h(k) \sum_{r=-\infty}^{\infty} h(r)\ \phi_{xx}(m+k-r)$$

Since the right hand side is independent of n, the system output y(n) is also independent of n. We can therefore write

$$\phi_{yy}(m) = \sum_{k=-\infty}^{\infty} h(k) \sum_{r=-\infty}^{\infty} h(r)\ \phi_{xx}(m+k-r)$$

Now let $\ell = r - k$ and rewrite the above summation can be written as

$$\phi_{yy}(m) = \sum_{\ell=-\infty}^{\infty} \phi_{xx}(m-\ell) \sum_{k=-\infty}^{\infty} h(k)\ h(\ell+k)$$

Now define

$$c_{hh}(\ell) = \sum_{k=-\infty}^{\infty} h(k)\ h(\ell+k) \qquad \text{(equation 2.188)}$$

We can now write $\phi_{yy}(m)$ as

$$\phi_{yy}(m) = \sum_{l=-\infty}^{\infty} \phi_{xx}(m-\ell) c_{hh}(\ell) \qquad \text{(equation 2.187)}$$

The sequence $c_{hh}(\ell)$ is an example of a <u>"deterministic" autocorrelation sequence</u>. (Since it is the autocorrelation of a <u>non-random</u> signal.)

Since the above equation is a convolution in the time domain, the frequency domain version this relation is

$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega}) \Phi_{xx}(e^{j\omega}) \qquad \text{(equation 2.189)}$$

where

$\Phi_{xx}(e^{j\omega})$ is the DTFT of $\phi_{xx}(m)$

$\Phi_{yy}(e^{j\omega})$ is the DTFT of $\phi_{yy}(m)$

and

$C_{hh}(e^{j\omega})$ is the DTFT of $c_{hh}(\ell)$.

$$C_{hh}(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} c_{hh}(\ell)e^{-j\omega\ell}$$

Using equation (2.188) to represent $c_{hh}(\ell)$, we can express $C_{hh}(e^{j\omega})$ as

$$C_{hh}(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h(k)\ h(\ell+k)e^{-j\omega\ell}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \sum_{\ell=-\infty}^{\infty} h(\ell+k)e^{-j\omega\ell}$$

Now let $m = \ell + k$:

$$C_{hh}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega(m-k)}$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{j\omega k} \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega m}$$

$$= H(e^{-j\omega})H(e^{j\omega}) = H^*(e^{j\omega})H(e^{j\omega}) = \ |H(e^{j\omega})|^2$$

Therefore, equation 2.189 can be written as

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\ \Phi_{xx}(e^{j\omega}) \qquad \text{(equation 2.190)}$$

Note that

$E\{y^2(n)\} = \phi_{yy}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})\, d\omega$ = <u>total average power</u> of the signal y(n)

$\Phi_{yy}(e^{j\omega})$ is called the <u>power density spectrum</u> of the random signal y(n). (It describes the contributions to the total average power in y(n) from its various component frequencies.)

<u>Note</u>: "total average power" is sometimes called "total power" or just "power."

<u>Example 2.26 White Noise</u>

If x(n) is a <u>white noise signal</u> then its autocorrelation is given by

$\phi_{xx}(m) = E\{x^2(n)\}\delta(m)$. If x(n) has zero-mean, then $\phi_{xx}(m) = \sigma_x^2\delta(m)$.

For this zero-mean case, the corresponding power density spectrum is

$\Phi_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}(m)e^{-j\omega m} = \sigma_x^2$ = constant, for all ω

<u>If a white noise signal is applied as the input to a linear time invariant system</u> having unit sample response h(n), the power density spectrum of the output y(n) is

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\, \Phi_{xx}(e^{j\omega})$$

$$= |H(e^{j\omega})|^2\, \sigma_x^2$$

For example, if a white noise signal is applied to a system having frequency response of

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

then:

$$\Phi_{yy}(e^{j\omega}) = \left|\frac{1}{1 - ae^{-j\omega}}\right|^2 \sigma_x^2 \quad = \quad \frac{\sigma_x^2}{1 + a^2 - 2a\cos\omega}$$

The cross-correlation between input and output of a LTI system whose input is a realization of a random process is given by

$$\phi_{xy}(m) = E\{x(n)y(n+m)\}$$

$$= E\left\{x(n)\sum_{k=-\infty}^{\infty} h(k)x(n+m-k)\right\}$$

$$= \sum_{k=-\infty}^{\infty} h(k)E\left\{x(n)x(n+m-k)\right\}$$

If x(n) is wide-sense stationary, then this can be written as

$$\phi_{xy}(m) = \sum_{k=-\infty}^{\infty} h(k)\phi_{xx}(m-k) \qquad \text{(equation 2.195)}$$

When the input is a white noise signal for which $\phi_{xx}(m) = \sigma_x^2\delta(m)$, the cross-correlation between input and output is:

$$\phi_{xy}(m) = \sum_{k=-\infty}^{\infty} h(k)\sigma_x^2\delta(m-k) = \sigma_x^2\ h(m) \qquad \text{(equation 2.197)}$$

The frequency domain version of equation 2.195 is

$\Phi_{xy}(e^{j\omega}) = H(e^{j\omega})\Phi_{xx}(e^{j\omega})$ (equation 2.196)

$\Phi_{xy}(e^{j\omega})$ is called the cross power spectrum of the system input and the system output.

As we have seen, the frequency domain version of $\phi_{xx}(m) = \sigma_x^2\delta(m)$ (white noise case) is

$\Phi_{xx}(e^{j\omega}) = \sigma_x^2$, for all ω

Therefore, for the case of a zero-mean white noise input, the cross power spectrum of the input and output is

$\Phi_{xy}(e^{j\omega}) = H(e^{j\omega})\sigma_x^2$ (equation 2.199)