ECE 8440 - Unit 6

Over-Sampling with A/D Conversion (see section 4.9)

Assume that the analog signal $x_a(t)$ is wide-sense stationary and that it is band-limited to $\Omega = \Omega_N$
That is,

$\Phi_{x_a x_a}(j\Omega) = 0, \quad |\Omega| \geq \Omega_N.$ (equation 4.158)

Consider the follow steps of processing this signal:

1. Sample $x_a(t)$ at a sampling rate that satisfies

   $$\frac{1}{T} > M\frac{\Omega_N}{\pi}.$$

Note that $\frac{\Omega_N}{\pi}$ is the Nyquist rate, and M is the over-sampling ratio. Call the sampled signal x(n).

2. Quantize the "over-sampled" signal x(n) to get $\hat{x}(n)$.
3. Apply an ideal low-pass digital filter with cutoff of $\omega_c = \frac{\pi}{M}$.
4. Down-sample the filter output by a factor of M.

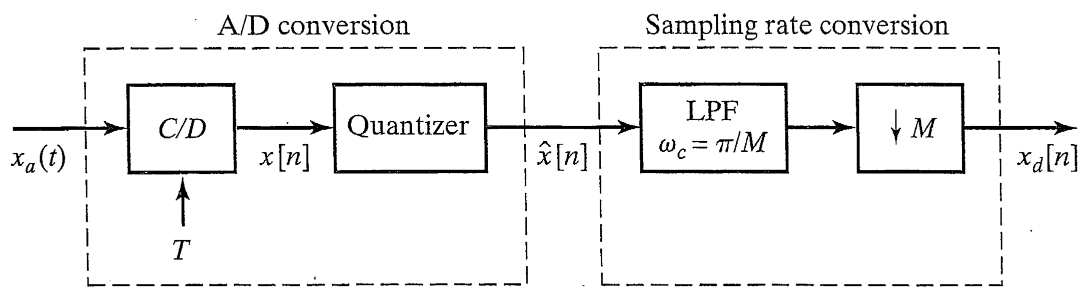

Figure 4.65 Oversampled A/D conversion with simple quantization and down-Sampling.

The quantization error can be treated as a source of additive white noise.

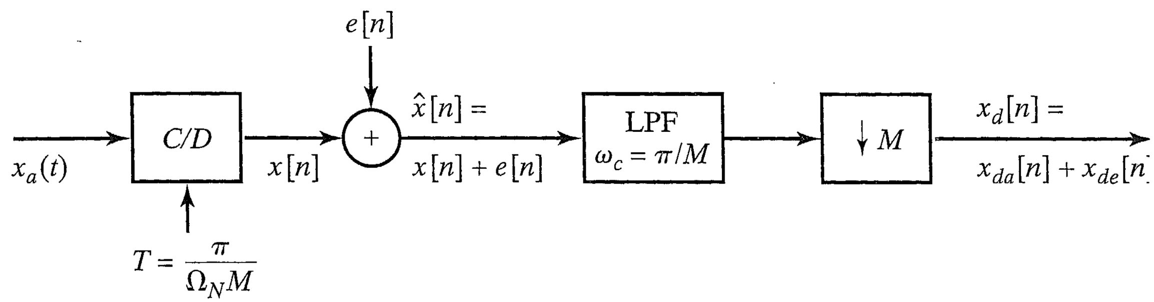

Figure 4.66 System of Figure 4.65 with quantizer replaced by linear noise model

Also note that the output of the above system consists of one component, $x_{da}(n)$, due to x(n) and one component, $x_{de}(n)$, due to e(n).

Goal: Determine ratio of signal power $E\left[x_{da}^2(n)\right]$ to the quantization noise power $E\left[x_{de}^2(n)\right]$, as a function of the quantization step size Δ and the over-sampling ratio M.

Effect of system of Figure 4.66 on the "signal component" x(n):

Note that

$$E\left[x(n+m)x(n)\right] = E\left[x_a((n+m)T)x_a(nT)\right]$$

Since we assume that $x_a(t)$ is wide-sense stationary, neither of the above expression is dependent on n, so we can write the above expression as:

$$\phi_{xx}(m) = \phi_{x_a x_a}(mT) \qquad \text{(equation 4.160)}$$

Note that $\phi_{xx}(m)$ can be considered to be a sampled version of $\phi_{x_a x_a}(t)$ with a sampling rate of $\frac{1}{T}$.

Also, note that

$$E[x^2(n)] = E[x_a^2(nT)].$$

Along with the assumption that $x_a(t)$ is wide-sense stationary, this indicates that the power in the original analog signal and the power in the sampled signal are the same, i.e.,

$$E[x^2(n)] = E[x_a^2(t)].$$

Note that $E[x^2(n)]$ and $E[x_a^2(t)]$ are related to their corresponding power spectral densities via

$$E[x_a^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{x_a x_a}(j\Omega)\, d\Omega = \frac{1}{2\pi}\int_{-\Omega_N}^{\Omega_N} \Phi_{x_a x_a}(j\Omega)\, d\Omega$$

and

$$E[x^2(n)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})\, d\omega = \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} \Phi_{xx}(e^{j\omega})\, d\omega.$$

The corresponding frequency domain expression of the relationship between $\phi_{xx}(m)$ and $\phi_{x_a x_a}(t)$ is (from the development of the Sampling Theorem):

$$\Phi_{xx}(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi_{x_a x_a}\left[j\left(\Omega - \frac{2\pi k}{T} \right) \right]. \qquad \text{(equation 4.162)}$$

Since we assumed that the input was band-limited to Ω_N and an over-sampling ratio of M was used, we can write:

$$\begin{aligned} \Phi_{xx}(e^{j\omega}) &= \frac{1}{T}\Phi_{x_a x_a}\left(j\frac{\omega}{T} \right), && |\omega| < \pi/M \\ &= 0, && \pi/M < |\omega| < \pi. \end{aligned}$$

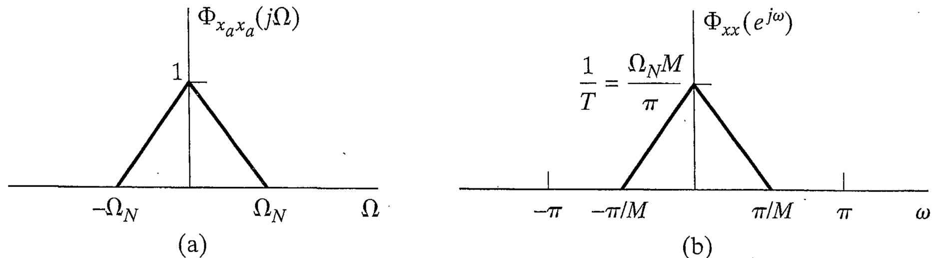

Figure 4.67 Illustration of frequency and amplitude scaling between $\Phi_{x_a x_a}(j\Omega)$ and $\Phi_{xx}(j\Omega)$.

The output of the down-sampler is $x_{da}(n) = x(nM)$.

Note that

$$E[x_{da}(n)x_{da}(n+m)] = E[x(nM)x((n+m)M)]$$

$$= E[x(nM)x(nM + mM)]$$

$$= \phi_{xx}(mM)$$

since $x_a(t)$, and therefore x(n), was assumed to be wide-sense stationary.

Therefore, we can write

$$\phi_{x_{da}x_{da}}(m) = \phi_{xx}(mM)$$

The frequency domain representation of this down-sampling relationship is

$$\Phi_{x_{da}x_{da}}(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} \Phi_{xx} e^{j(\omega-2\pi k)/M}.$$

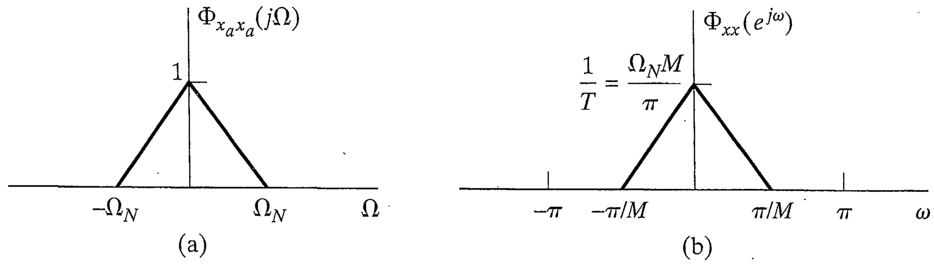

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Figure 4.67 Illustration of frequency and amplitude scaling between

Since x(n) is bandlimited to π/M, there is no aliasing due to down-sampling, and one period of $\Phi_{x_{da}x_{da}}(e^{j\omega})$ can be expressed as

$$\Phi_{x_{da}x_{da}}(e^{j\omega}) = \frac{1}{M}\Phi_{xx}\left(e^{j\omega/M}\right), \qquad |\omega| < \pi.$$

We can now <u>determine the power</u> of $x_{da}(n)$ as follows:

$$E[x_{da}^2(n)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{x_{da}x_{da}}(e^{j\omega})\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{M}\Phi_{xx}(e^{j\omega/M})\,d\omega.$$

If we let $\omega' = \omega/M$, the above integral can be written as

$$= \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} \frac{1}{M}\Phi_{xx}(e^{j\omega'})\, M d\omega'$$

$$= \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} \Phi_{xx}(e^{j\omega'})\, d\omega' \quad = \quad E[x^2(n)].$$

This shows that the power of the output $\mathbf{x_{da}(n)}$ of the system of Figure 4.56 (the part of the output due to **x(n)**) is the same the power in **x(n)**, which has already been shown to have the same power as the system input $\mathbf{x_a(t)}$.

Quantization noise component of system output

As before, assume that the injected quantization noise e(n) is a wide-sense stationary, white noise process with zero mean and the following variance:

$$\sigma_e^2 = \frac{\Delta^2}{12}.$$

As already shown, the corresponding autocorrelation is

$$\phi_{ee}(m) = \sigma_e^2\, \delta(m)$$

and the power density spectrum is

$$\Phi_{ee}(e^{j\omega}) = \sigma_e^2\,, \quad |\omega| < \pi \quad \text{where} \quad \sigma_e^2 = \frac{\Delta^2}{12}.$$

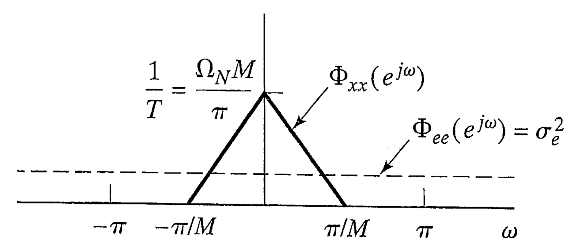

Figure 4.68 Power spectral density and quantization noise with an oversampling factor on M.

Referring to equation 2.190, the contribution to the power density spectrum of the output of the low-pass filter (with cutoff $\omega_c = \pi / M$ and gain = 1), due to an input of e(n) , is

$$\Phi_{lpf-e}(e^{j\omega}) = \Phi_{ee}(e^{j\omega}) \,|\, H(e^{j\omega}) \,|^2$$

$$= \Phi_{ee}(e^{j\omega}) \cdot 1 = \sigma_e^2 \qquad \text{for } |\omega| \le \pi/M$$

$$= 0 \qquad \text{for } \pi/M \le |\omega| \le \pi.$$

The low-pass filter in Figure 4.66 removes frequency components of e(n) in the range $(\omega_c = \pi / M) \le |\omega| \le \pi$. The noise power at the output of this low-pass filter is:

$$E\{e_n^2(n)\} = \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \sigma_e^2 \, d\omega = \frac{\sigma_e^2}{2\pi}\left[\frac{2\pi}{M}\right] = \frac{\sigma_e^2}{M}.$$

Finally, down-sampling by a factor of M applies a 1/M magnitude scale factor to $\Phi_{lpf-e}(e^{j\omega})$ and increases the upper frequency from π/M to π, as shown in Figure 4.60.

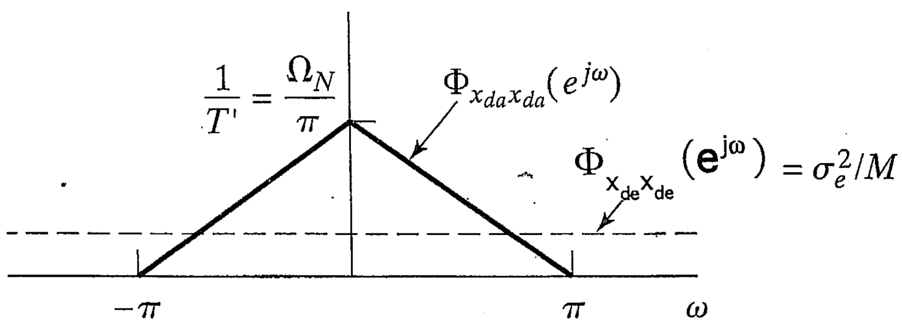

Figure 4.69 Power spectral density of signal and quantization noise after down-sampling

As we saw for the case of the signal component x(n), passing a signal through a down-sampler does not change the power of the input signal. Therefore, the contribution of e(n) to the power of the system output is the same as the contribution of e(n) to the power of the output of the low-pass filter, which was shown to be $\frac{\sigma_e^2}{M}$.

We can confirm this by calculating $E\{x_{de}^2\}$ directly by evaluating the IDTFT of $\Phi_{x_{de}x_{de}}(e^{j\omega})$ for the m = 0 case:

$$E\{x_{de}^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_e^2}{M}\,\mathrm{d}\omega = \frac{\sigma_e^2}{M} = \frac{\Delta^2}{12M}. \qquad \text{(equation 4.169)}$$

<u>Investigate the trade-off between M and Δ.</u>

Recall that Δ is related to the B (the number of quantizer output bits used to represent the positive range of the input) and X_m (the input can range from - X_m to X_m) is

$$\Delta = \frac{X_m}{2^B}.$$

Therefore, the output power of noise due to quantization can be expressed as

$$E\{x_{de}^2\} = \frac{1}{12M}\left(\frac{X_m}{2^B}\right)^2.$$

For fixed quantizer parameters (X_m and B), the noise power can be decreased by increasing the over-sampling factor M.

Since the power output due to the "good signal" is independent of M, decreasing $E\{x_{de}^2\}$ also increases the signal-to-quantization-noise ratio (SNR).

We can solve for the number of quantizer bits needed to achieve a target value P_{de} = $E\{x_{de}^2\}$ as follows:

$$12MP_{de}2^{2B} = X_m^2$$

$$2^{2B} = \frac{X_m^2}{12MP_{de}}$$

$$2B = \log_2 X_m^2 - \log_2 12MP_{de}$$

$$= 2\log_2 X_m - \log_2 12MP_{de}$$

$$B = \log_2 X_m - \frac{1}{2}\log_2 12 - \frac{1}{2}\log_2 M - \frac{1}{2}\log_2 P_{de}.$$

If the oversampling ratio M is replaced by $M' = KM$ (while keeping P_{de} and X_m fixed), the new value for the number of quantizer bits required is

$$B' = \log_2 X_m - \frac{1}{2}\log_2 12 - \frac{1}{2}\log_2 KM - \frac{1}{2}\log_2 P_{de}$$

$$= \log_2 X_m - \frac{1}{2}\log_2 12 - \frac{1}{2}\log_2 M - \frac{1}{2}\log_2 K - \frac{1}{2}\log_2 P_{de}.$$

Therefore, to decrease B' by 1 (while keeping P_{de} and X_m fixed), K must be chosen to satisfy

$$-\frac{1}{2}\log_2 K = -1$$

so that

$\log_2 K = 2$ and $K = 4$.

<u>Another Example</u>:

To decrease the number of quantizer bits B from 16 to 12 (decrease of 4 bits), while keeping P_{de} and X_m fixed), the oversampling ratio M would have to be increased by a factor of K which satisfies

$$-\frac{1}{2}\log_2 K = -4$$

$$\log_2 K = 8$$

$$K = 256 \; .$$