ECE 8440 Unit 7

Using Noise Shaping to Enhance Over-sampled A/D Conversion (see section 4.9.2)

General approach: Modify the over-sampled A/D conversion process so that more of the noise power is outside the pass-band of the low-pass filter used in figure 4.66. (That is, more of the noise is outside the range defined by $|\omega| < \pi/M$.)

Consider the quantizer shown in the figure below:

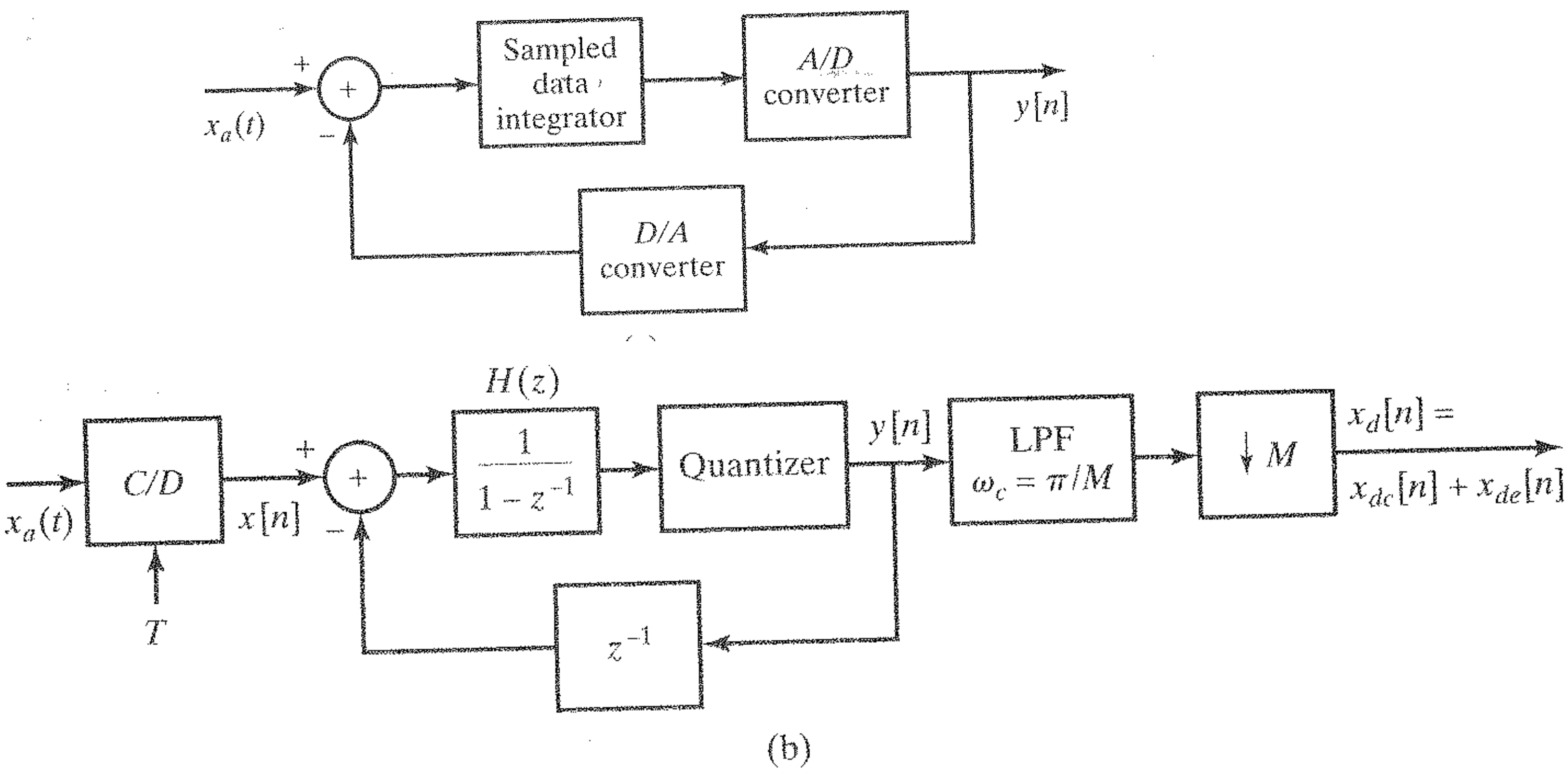


Figure 4.70 Oversampled quantizer with noise shaping.

As before, we can model the effect of the quantizer by replacing the quantizer with a summation node that adds quantization noise to the quantizer input.

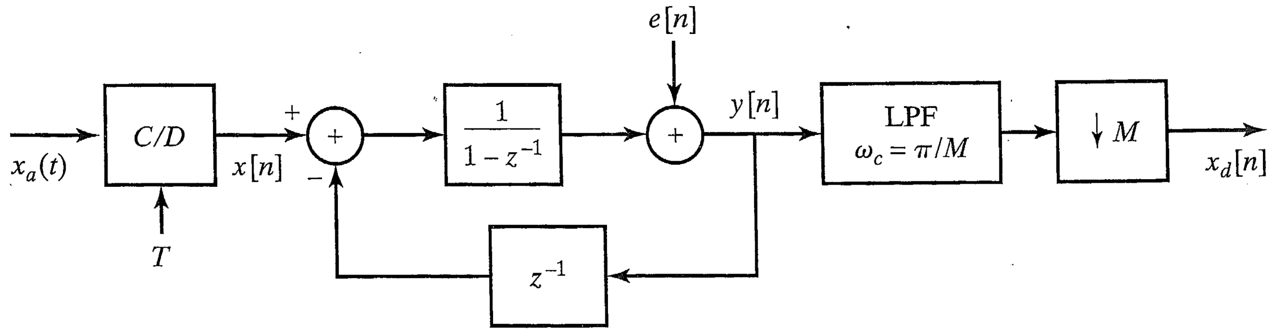


Figure 4.71 System of Figure 4.70 from $x_a(t)$ to $x_d(n)$ with quantizer replaced by a linear noise model.

In the above figure, the quantizer output **y(n)** is the sum of two contributions:

y_x(n) , due to the quantizer input **x(n)** alone

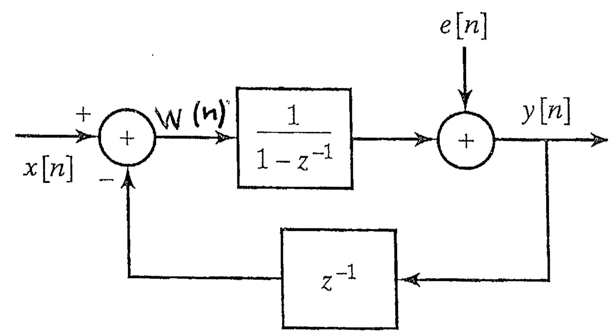
$\hat{e}$(n) , due to the quantization noise **e(n)** alone.

<u>First, determine the transfer function from **x(n)** to **y(n)**</u> (call this **H_x(z)**)

To find **H_x(z)**, first denote the output of the first summation node as **w(n)**. For this analysis, assume that **e(n) =0**.

$w(n) = x(n) - y(n-1)$

$W(z) = X(z) - Y(z)z^{-1}$



Also from the figure we can see that

$$Y(z) = W(z)\left(\frac{1}{1-z^{-1}}\right).$$

Combining the above two expressions gives

$$Y(z) = \left(X(z) - z^{-1}Y(z)\right)\left(\frac{1}{1-z^{-1}}\right)$$

$$Y(z) - z^{-1}Y(z) = X(z) - z^{-1}Y(z)$$

$$Y(z) = X(z).$$

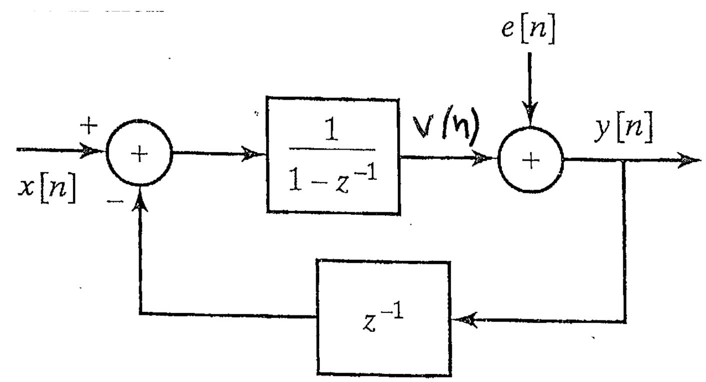
Therefore, $\mathbf{H_x(z) = 1}$

<u>Now determine the transfer function from **e(n)** to **y(n)**</u> (call this $\mathbf{H_e(z)}$). For this analysis, assume that **x(n) = 0**.

Denote as **v(n)** the other input to the summation node that **e(n)** feeds. The output of this summation node due to quantization noise is then

$$y(n) = e(n) + v(n)$$

$$Y(z) = E(z) + V(z)$$



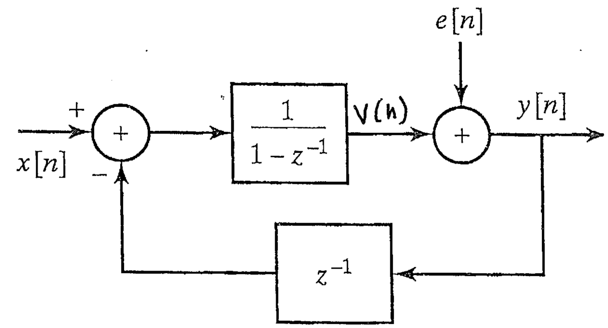
Also, since we are temporarily assuming that **x(n) = 0**, the relation between **V(z)** and **Y(z)** is

$$V(z) = Y(z)\left[-z^{-1}\frac{1}{1-z^{-1}}\right]$$

Combining the previous two expressions gives

$$Y(z) = E(z) - Y(z)\left[z^{-1}\frac{1}{1-z^{-1}}\right]$$

$$E(z) = Y(z)\left[1+\frac{z^{-1}}{1-z^{-1}}\right] = Y(z)\left[\frac{1-z^{-1}+z^{-1}}{1-z^{-1}}\right] = Y(z)\left[\frac{1}{1-z^{-1}}\right]$$



Therefore,

$$\frac{Y(z)}{E(z)} = 1 - z^{-1} = H_e(z).$$

The contribution to the output $y(n)$ from the quantization error $e(n)$ is therefore

$\hat{e}(n) = e(n) - e(n-1).$

Also, since $H_x(z) = 1$, the contribution to $y(n)$ from $x(n)$ is

$y_x(n) = x(n).$

We can therefore re-draw figure 4.71 as

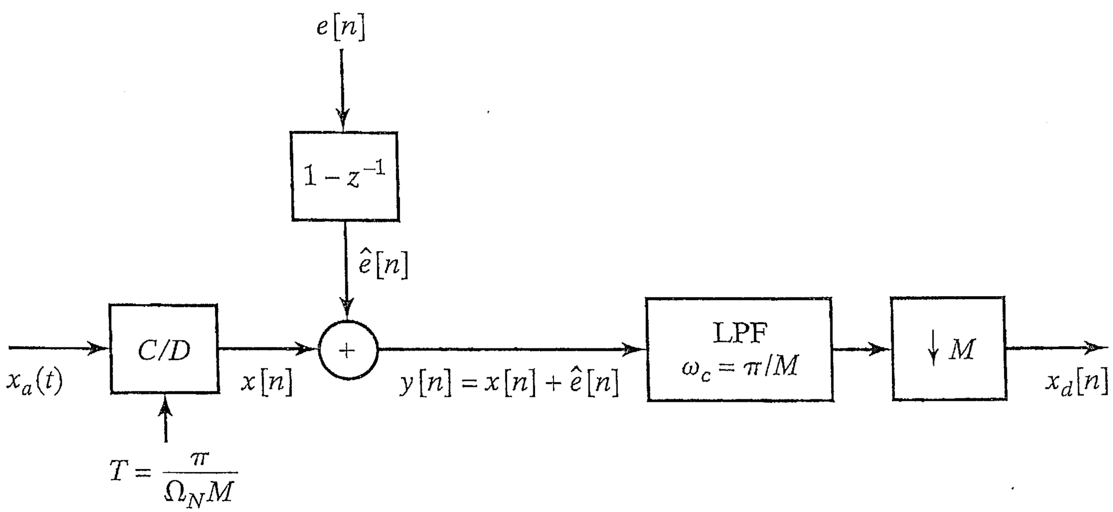


Figure 4.72 Equivalent representation of Figure 4.71

The <u>power density spectrum</u> of the quantization noise $\hat{e}(n)$ that is present in y(n) is

$$\Phi_{\hat{e}\hat{e}}(e^{j\omega}) = \sigma_e^2 \,|H_e(e^{j\omega})|^2 .$$

Note that

$$H_e(e^{j\omega}) = (1 - z^{-1})|_{z=e^{j\omega}}$$

$$= (1 - e^{-j\omega}) = e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right] = e^{-j\frac{\omega}{2}} 2j\sin(\omega/2)$$

So

$$|H_e(e^{j\omega})| = 2\,|\sin(\omega/2)|$$

and

$$\Phi_{\hat{e}\hat{e}}(e^{j\omega}) = \sigma_e^2\left[2\sin(\omega/2)\right]^2$$

$$= 4\sigma_e^2 \sin^2(\omega/2) \qquad \text{(equation 4.174)}$$

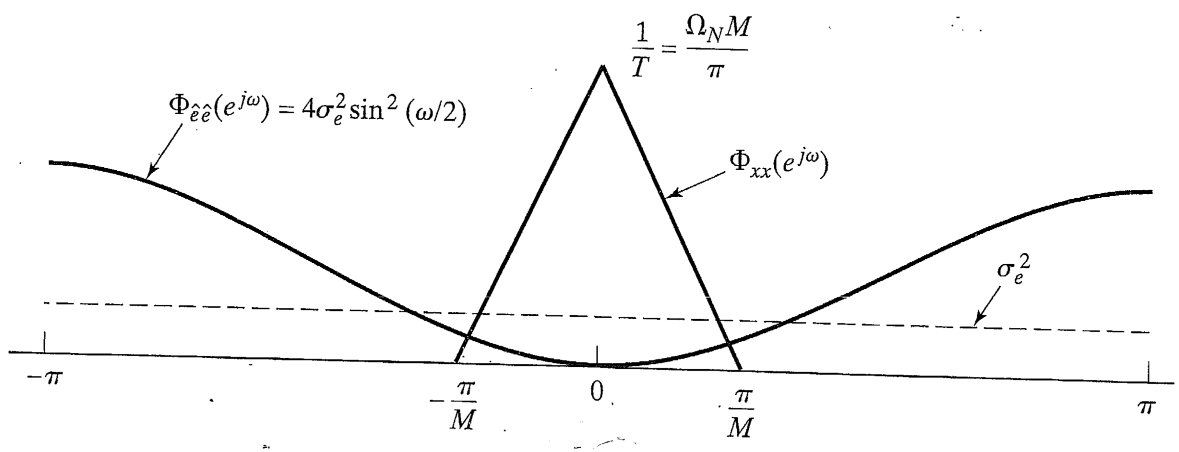


Figure 4.73 The power spectral density of the quantization noise and the signal.

The noise power of $\hat{e}(n)$ is

$$
\begin{aligned}
E\{\hat{e}^2(n)\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\hat{e}\hat{e}}(e^{j\omega})d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sigma_e^2 \sin^2(\omega/2)d\omega \\
&= \frac{2}{\pi}\sigma_e^2\int_{-\pi}^{\pi} \sin^2(\omega/2)d\omega \\
&= \frac{2}{\pi}\sigma_e^2\int_{-\pi}^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos(\omega)\right)d\omega \quad = \frac{2}{\pi}\sigma_e^2\left[\frac{1}{2}(2\pi)-\frac{1}{2}\sin(\omega)\Big|_{-\pi}^{\pi}\right] \\
&= \frac{2}{\pi}\sigma_e^2[\pi-0] = 2\sigma_e^2.
\end{aligned}
$$

Although this is twice the size of $E\{e^2(n)\}=\sigma_e^2$, the quantization noise has been shaped so that less of the noise is in the frequency band $(|\omega|<\pi/M)$ occupied by the over-sampled signal, as shown in the figure on the previous slide.

After passing though the low-pass filter with cutoff of π/M and down-sampling by a factor of M (the last two boxes in fig. 4.71) the power spectral density of the quantization noise, $\Phi_{x_{de}x_{de}}(e^{j\omega})$, and of the signal, $\Phi_{x_{da}x_{da}}(e^{j\omega})$, are as shown on the next slide.

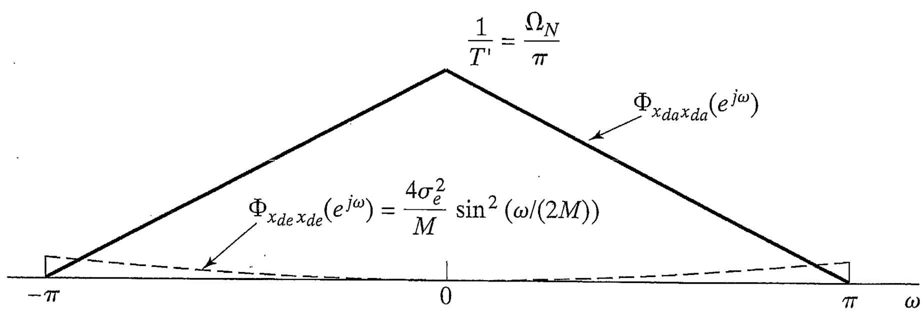


Figure 4.74 Power spectral density of the signal and quantization noise after downsampling.

As seen in the above figure, we can express the power spectral density of the quantization noise after low-pass filtering and downsampling, over the frequency range $|\omega| < \pi$, in terms of the power spectral density of the quantization noise before low-pass filtering and down-sampling as

$$\Phi_{x_{de}x_{de}}(e^{j\omega}) = \frac{1}{M}\Phi_{\hat{e}\hat{e}}(e^{j\frac{\omega}{M}}), \qquad |\omega| < \pi$$

$$= \frac{4}{M}\sigma_e^2 \sin^2(\omega/2M)$$

The quantization noise power in the output can then be obtained as

$$E\{x_{de}^2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{x_{de}x_{de}}(e^{j\omega})d\omega$$

$$= \frac{4\sigma_e^2}{2\pi M}\int_{-\pi}^{\pi}\sin^2(\omega/2M)\,d\omega.$$

Since $\sigma_e^2 = \Delta^2/12$ the above can be written as

$$E\{x_{de}^2(n)\} = \frac{4\Delta^2}{(12)2\pi M}\int_{-\pi}^{\pi} \sin^2(\omega/2M)\,d\omega$$

$$= \frac{\Delta^2}{6\pi M}\int_{-\pi}^{\pi} \sin^2(\omega/2M)\,d\omega.$$

<u>To obtain an approximate value</u> of $E\{x_{de}^2(n)\}$, assume that M is sufficiently large that

$\sin(\omega/2M) \simeq \omega/2M.$

Then the above expression for $E\{x_{de}^2(n)\}$ can be approximated as

$$= \frac{\Delta^2}{6\pi M}\frac{1}{4M^2}\int_{-\pi}^{\pi}\omega^2 d\omega = \frac{\Delta^2}{24\pi M^3}\frac{\omega^3}{3}\Bigg|_{-\pi}^{\pi} = \frac{\Delta^2}{72\pi M^3}\left(\pi^3 - (-\pi)^3\right)$$

$$= \frac{\Delta^2 2\pi^3}{72\pi M^3} = \frac{\Delta^2\pi^2}{36M^3} = E\{x_{de}^2(n)\} \triangleq P_{de}. \qquad \text{(equation 4.176)}$$

$$= \sigma_e^2 \underbrace{\left(\frac{\pi^2}{3M^3}\right)}_{\substack{\text{approximate}\\ \text{reduction factor}\\ \text{for large M}}}, \qquad \text{since } \sigma_e^2 = \Delta^2/12.$$

<u>To investigate the trade-off between M and B</u> in controlling P_{de} we can substitute $\Delta = X_m/2^B$ in equation 4.176 to get:

$$P_{de} = \frac{X_m^2\pi^2}{2^{2B}36M^3}.$$

Now solve for B in terms of the other variables:

$$2^{2B} = \frac{X_m^2 \pi^2}{36M^3 P_{de}}$$

$$2B = 2\log_2(X_m) + 2\log_2(\pi) - \log_2(36) - 3\log_2(M) - \log(P_{de})$$

$$B = \log_2(X_m) + \log_2(\pi) - \frac{1}{2}\log_2(36) - \frac{3}{2}\log_2(M) - \frac{1}{2}\log(P_{de})$$

$$= \log_2(X_m) + \log_2(\pi) - \log_2(6) - \frac{3}{2}\log_2(M) - \frac{1}{2}\log(P_{de})$$

$$= \log_2(X_m) + \log_2(\frac{\pi}{6}) - \frac{3}{2}\log_2(M) - \frac{1}{2}\log(P_{de}). \qquad \text{(equation 4.177)}$$

Note that doubling the over-sampling ratio M reduces the number of bits, B, required to attain a target P_{de} value by 3/2 (assuming that X_m is unchanged).

Recall that without noise shaping, doubling the over-sampling ratio M only reduced the number of bits, B, required to attain a target P_{de} value by 1/2.

The table below shows the number of quantizer bits that can be reduced (and still maintain a fixed P_{de} value) by using an over-sampling ratio M with an A/D converter, both with and without noise shaping.

M	Direct quantization	Noise shaping
4	1	2.2
8	1.5	3.7
16	2	5.1
32	2.5	6.6
64	3	8.1

TABLE 4.1 EQUIVALENT SAVINGS IN QUANTIZER BITS RELATIVE TO $M = 1$ FOR DIRECT QUANTIZATION AND FIRST-ORDER NOISE SHAPING

The above noise shaping approach can be extended by using multiple stages, as shown in the figure below:

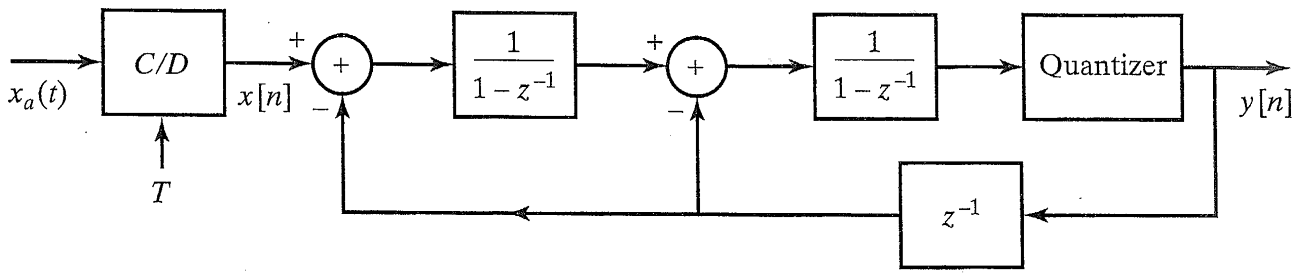


Figure 4.75 Oversampled quantizer with second-order noise shaping.

For this two-stage system, the transfer function between the quantization noise source and the output is

$H_e(z) = (1 - z^{-1})^2$ (the square of H(z) for one-stage of noise shaping).

Therefore, the power density spectrum of quantization noise at the system output can be found by squaring the system contribution in the one-stage system.

Before: $\Phi_{\hat{e}\hat{e}}(e^{j\omega}) = \sigma_e^2 \left[2\sin(\omega/2)\right]^2$ (one-stage noise shaping)

Now: $\Phi_{\hat{e}\hat{e}}(e^{j\omega}) = \sigma_e^2 \left[2\sin(\omega/2)\right]^4$ (two-stage noise shaping)

<u>For p stages of noise shaping</u>, the corresponding power density spectrum of quantization noise at the system output can be expressed as

$$\Phi_{\hat{e}\hat{e}}(e^{j\omega}) = \sigma_e^2 \left[2 \sin(\omega / 2)\right]^{2p} .$$

The table below shows how the number of quantizer bits can be reduced (and still maintain a fixed value of P_{de}) for several values of p and M.

9

TABLE 4.2 REDUCTION IN QUANTIZER BITS AS ORDER p OF NOISE SHAPING

Noise shaping ~~Quantizer~~ order p	Oversampling factor M				
	4	8	16	32	64
0	1.0	1.5	2.0	2.5	3.0
1	2.2	3.7	5.1	6.6	8.1
2	2.9	5.4	7.9	10.4	12.9
3	3.5	7.0	10.5	14.0	17.5
4	4.1	8.5	13.0	17.5	22.0
5	4.6	10.0	15.5	21.0	26.5

} Same info. as in Table 4.1. (rows 0–1)

<u>Example:</u>

With p = 2 and M = 64 the number of quantizer bits could be reduced by about 12.9 relative to the case of not using any over-sampling. Therefore, a 1-bit quantizer with p = 2 and M = 64 would contribute the same quantization noise power to the output as a 14-bit quantizer without over-sampling.

<u>Note:</u> For large values of p, there is a risk of instability and oscillations to occur.