Use of Over-Sampling and Noise Shaping in D/A Conversion

If we implement over-sampling with noise shaping in the D/A conversion process, we can reduce the number of bits used in the D/A converter (e.g., 16 bits → 8 bits), without introducing a major amount of quantization noise to the output. The figure below shows the overall structure of the proposed system.

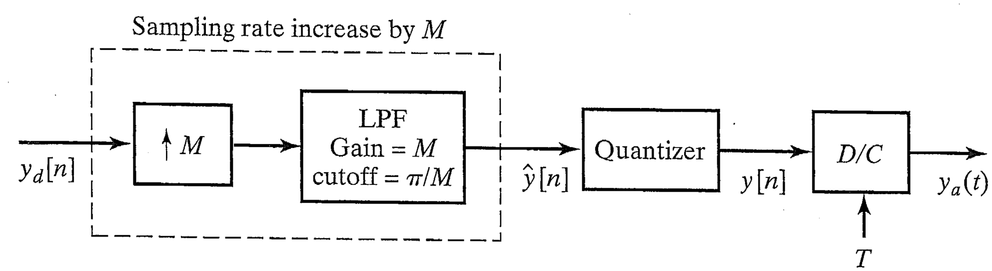


Figure 4.76 Oversampled D/A conversion

The figure below shows the use of a first-order noise shaping system used to help implement the quantizer shown in the above figure:

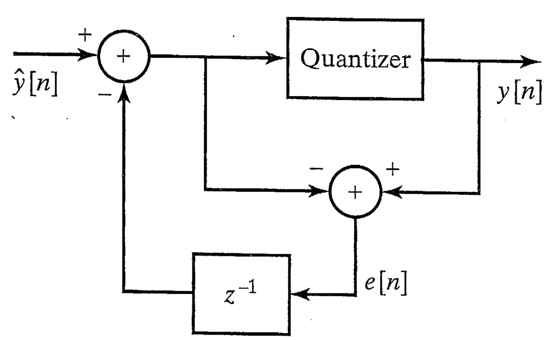


Figure 4.77 First-order noise-shaping system for oversampled D.A quantization.

As before, the quantizer in the previous diagram can be represented by an additive source of white noise, e(n), as shown below:

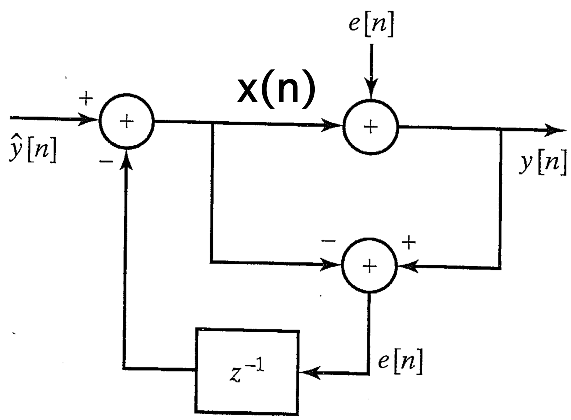


Figure 4.78 System of Figure 4.77 with quantizer replaced by linear noise model.

Since $\quad y(n) = x(n) + e(n)$

then $\quad e(n) = y(n) - x(n)$.

<u>First, determine the transfer function from $\hat{y}(n)$ to the quantizer output **y(n)**</u> in figure 4.78, with the **e(n)** contribution set to zero. As can be seen from this figure,

$y(n) = \hat{y}(n)$ $\qquad$ (assuming e(n) = 0).

The transfer function from $\hat{y}(n)$ to the quantizer output $y(n)$ is therefore

$$H_{\hat{y}}(z) = 1$$

Next, determine the transfer function from the quantization error e(n) to the quantizer output **y(n)**, with $\hat{y}(n)$ set to zero. As can be seen from figure 4.78,

y(n) = e(n) - e(n-1)

The transfer function from e(n) to y(n) is therefore

$$H_e(z) = 1 - z^{-1}$$

As already shown for the case of C/D conversion, the magnitude of the frequency response of a filter with this transfer function is

$$|H_e(e^{j\omega})| = 2\,|\sin(\omega/2)|$$

Therefore, the output power density spectrum due to the noise signal e(n) is

$$\Phi_{\hat{e}\hat{e}}(e^{j\omega}) = \sigma_e^2 (2\sin(\omega/2))^2 \qquad \text{(equation 4.181)}$$

Example of figure 4.79

Let $\mathbf{y_d(n)}$ denote a signal which is the output of a digital filter and which is to be converted back to a continuous-time signal. (It is assumed that the Nyquist sampling rate is being used in representing $\mathbf{y_d(n)}$.)

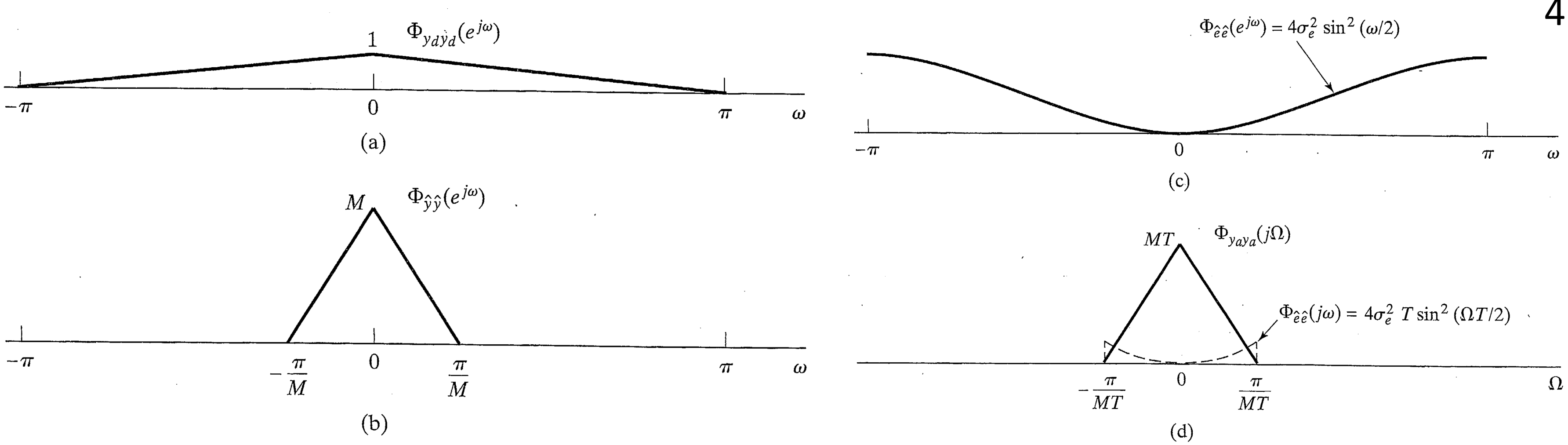


Figure 4.79 (a) Power spectral density of signal $y_d(n)$ (b) Power spectral density of signal $\hat{y}(n)$ (c) Power spectral density of quantization noise. (d) Power spectral density of the continuous-time signal and the quantization noise.

<u>In part "d" of the above figure</u>: The final step in the process is a <u>modified</u> ideal D/C converter which uses a conversion rate of $\frac{1}{\mathbf{T_{out}}}$ and a low-pass analog filter with cutoff of $\left(\frac{\pi}{\mathbf{M}}\right)\frac{1}{\mathbf{T_{out}}}$ instead of the normal cutoff of $\left(\frac{\pi}{\mathbf{T_{out}}}\right)$. The gain of the lowpass filter is $\mathbf{T_{out}}$, which is the same as for the "original" ideal D/C converter.

Since it was assumed that the Nyquist rate was used to sample the original analog input, we know that input sampling rate $\frac{\mathbf{1}}{\mathbf{T_{in}}}$ satisfies

$\frac{\mathbf{1}}{\mathbf{T_{in}}} = \frac{\mathbf{\Omega_{max}}}{\pi}$ which corresponds to $\mathbf{\Omega_{max}} = \frac{\pi}{\mathbf{T_{in}}}$.

Because of the intermediate step of up-sampling by a factor of M , the D/C converter at the output will be operating at a sample interval of $\mathbf{T}_{out} = \frac{\mathbf{T}_{in}}{\mathbf{M}}$

Therefore, the highest frequency in the analog output will be

$$\mathbf{\Omega}_{out} = \left(\frac{\pi}{\mathbf{M}}\right)\frac{1}{\mathbf{T}_{out}} = \left(\frac{\pi}{\mathbf{M}}\right)\frac{\mathbf{M}}{\mathbf{T}_{in}} = \frac{\pi}{\mathbf{T}_{in}} = \mathbf{\Omega}_{max}$$, as desired.

Note: The above development assumes using an ideal analog low-pass filter in the D/C converter. If more D/A quantization noise can be tolerated in the output, then a non-ideal analog filter can be used. (If multi-stage noise shaping is used, then a non-ideal analog filter can be used with less penalty in overall performance, since more of the D/A quantization noise is pushed to higher frequencies, farther from the cutoff frequency of the analog filter.)

Phase-Related Topics from Chapter 5

Consider the system function for a system that has N poles and M zeros.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$ (equation 5.20)

H(z) can also be represented in factored form as follows:

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$ (equation 5.21)

The zeros of H(z) are at located at the $\{c_k\}$ and the poles of H(z) are at the $\{d_k\}$.

The frequency response for the system whose H(z) can be found from:

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

If H(z) is represented in the factored form as in equation 5.21 above, its frequency response can be expressed as:

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})} \qquad \text{(equation 5.46)}$$

In general, $H(e^{j\omega})$ can be expressed in terms of a magnitude response function and a phase response:

$$H(e^{j\omega}) = \;|H(e^{j\omega})|\,e^{j\measuredangle H(e^{j\omega})} \qquad \text{or} \qquad H(e^{j\omega}) = \;|H(e^{j\omega})|\,e^{j\arg[H(e^{j\omega})]}$$

Starting with the representation of H(z) in equation 5.21, the phase response function can be written as

$$\arg H(e^{j\omega}) = \arg\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M}\arg[1-c_k e^{-j\omega}] - \sum_{k=1}^{N}\arg[1-d_k e^{-j\omega}] \qquad \text{(equation 5.51)}$$

Note that $\arg H(e^{j\omega})$ and $\arg H(e^{j\omega}) + 2\pi r$, where r is an integer, contribute the same to $H(e^{j\omega})$.

That is because

$$e^{j[\arg H(e^{j\omega})+2\pi r]} = e^{j[\arg H(e^{j\omega})]} \underbrace{e^{j[2\pi r]}}_{1}$$

If r is selected so that $\arg H(e^{j\omega}) + 2\pi r$ is between -π and π, the resulting sum is called the Principal Value of phase and is denoted ARG $H(e^{j\omega})$. That is, $-\pi < \mathrm{ARG}[H(e^{j\omega})] < \pi$

If the phase response function is not limited to this range, it is denoted as $\arg[H(e^{j\omega})]$.

The following figure shows a typical relationship between ARG[H($e^{j\omega}$)] and arg[H($e^{j\omega}$)].H($e^{j\omega}$)

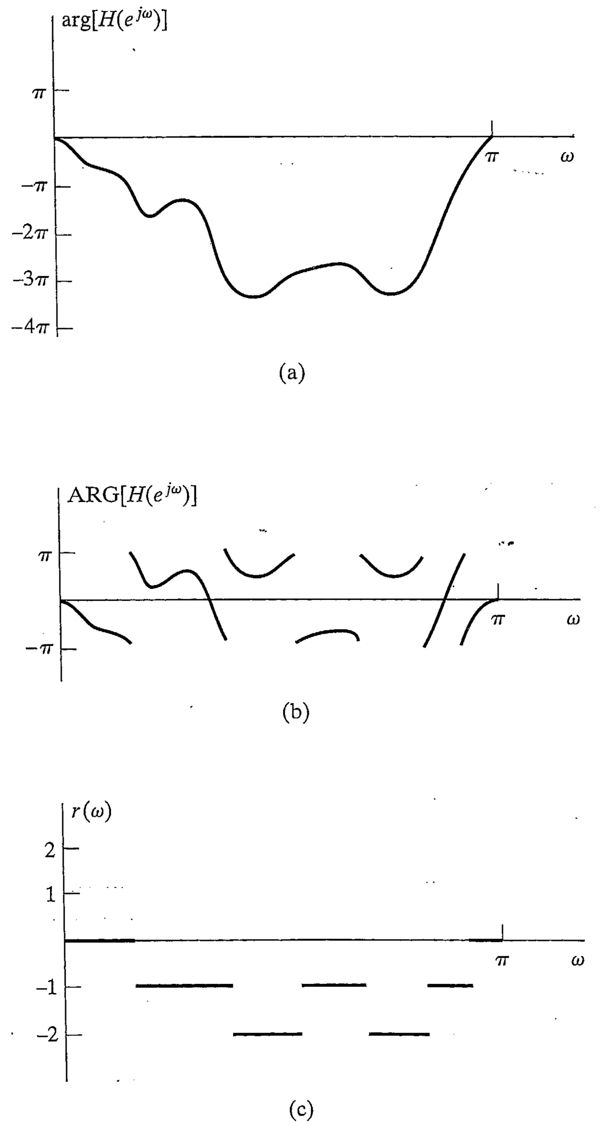


Figure 5.1 (a) Continuous-phase curve for a system function evaluated on the unit circle. (b) Principal value of the phase curve in part (a). (c) Integer multiples of 2π to be added to ARG[H($e^{j\omega}$)] to obtain arg[H($e^{j\omega}$)]

The group delay of a system is defined as

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}[\arg[H(e^{j\omega})]$$

Note that except for values of ω for which $\text{ARG}[H(e^{j\omega})]$ has a discontinuity,

$$\frac{d}{d\omega}[\text{ARG}[H(e^{j\omega})] = \frac{d}{d\omega}[\arg[H(e^{j\omega})]$$

Based on the representation of the phase response function in equation 5.51. we can write

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N}\frac{d}{d\omega}(\arg[1-d_k e^{-j\omega}) - \sum_{k=1}^{M}\frac{d}{d\omega}(\arg[1-c_k e^{-j\omega}) \qquad \text{(equation 5.52)}$$

Using the fact that

$$\arg[H(e^{j\omega}) = \arctan\left[\frac{\text{Im}[H(e^{j\omega})]}{\text{Re}[H(e^{j\omega})]}\right]$$

and taking derivatives, the group delay can be expressed as

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N}\frac{|d_k|^2 - \text{Re}\{d_k e^{-j\omega}\}}{1+|d_k|^2 - 2\text{Re}\{d_k e^{-j\omega}\}} - \sum_{k=1}^{M}\frac{|c_k|^2 - \text{Re}\{c_k e^{-j\omega}\}}{1+|c_k|^2 - 2\text{Re}\{c_k e^{-j\omega}\}}. \qquad \text{(equation 5.53)}$$

as will be verified later in this unit.

Phase response of a single zero or pole

Consider the phase response of one of the terms in the factored form of $H(e^{j\omega})$ shown in equation 5.46 with the zero or pole represented as $\mathbf{c}_k$ or $\mathbf{d}_k = re^{j\theta}$:

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \arg[1 - r\cos(\theta-\omega) - jr\sin(\theta-\omega)]$$

$$= \arctan\left[\frac{-r\sin(\theta-\omega)}{1 - r\cos(\theta-\omega)}\right] = \arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right].$$

Then we can determine the group delay of this term as follows.

$$-\frac{d}{d\omega}\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

$$= -\frac{1}{1+\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]^2}\cdot\frac{d}{d\omega}\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

Note that

$$\frac{d}{d\omega}\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

$$= \frac{\left[1 - r\cos(\omega-\theta)\right]\left[r\cos(\omega-\theta)\right] - \left[r\sin(\omega-\theta)\right]\left[r\sin(\omega-\theta)\right]}{\left[1 - r\cos(\omega-\theta)\right]^2}$$

$$= \frac{r\cos(\omega-\theta) - r^2\cos^2(\omega-\theta) - r^2\sin^2(\omega-\theta)}{\left[1 - r\cos(\omega-\theta)\right]^2}$$

$$= \frac{r\cos(\omega-\theta) - r^2}{\left[1 - r\cos(\omega-\theta)\right]^2}$$

Therefore, the group delay is

$$= -\frac{1}{1+\left[\dfrac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]^2} \cdot \frac{r\cos(\omega-\theta)-r^2}{\left[1-r\cos(\omega-\theta)\right]^2}$$

$$= \frac{r^2 - r\cos(\omega-\theta)}{\left[1-r\cos(\omega-\theta)\right]^2 + r^2\sin^2(\omega-\theta)} = \frac{r^2 - r\cos(\omega-\theta)}{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)+r^2\sin^2(\omega-\theta)}$$

which can be written as

$$= \frac{r^2 - r\cos(\omega-\theta)}{1-2r\cos(\omega-\theta)+r^2} = \text{grd}\left[1-re^{j\theta}e^{-j\omega}\right].$$

For a system having an arbitrary, finite number of poles and zeros, the above results for the phase and group delay can be easily extended. For example, if a system includes a pair of complex conjugate poles at $z = re^{j\theta}$ and $z = re^{-j\theta}$, H(z) can be expressed as

$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

Then, using results developed above, the phase response for the above second order term is

$$\sphericalangle H(e^{j\omega}) = -\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] - \arctan\left[\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)}\right] \qquad \text{(equation. 5.63b)}$$

and the corresponding group delay is

$$\text{grd}[H(e^{j\omega})] = -\frac{r^2 - r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)} - \frac{r^2 - r\cos(\omega+\theta)}{1+r^2-2r\cos(\omega+\theta)} \qquad \text{(equation 5.63c)}$$

which is consistent with equation 5.53 shown earlier.

Relationship Between Magnitude and Phase (Section 5.4)

In general: the magnitude response and the phase response of a system are independent.

For a rational system: For a given magnitude response, there are multiple possible phase responses. If the number of poles and zeros is known, the number of possible phase responses is finite. Otherwise, it is infinite. Show this:

Given some $|H(e^{j\omega})|$, we can form $|H(e^{j\omega})|^2$ which can be expressed as $|H(e^{j\omega})|^2 = H(e^{j\omega})\, H^*(e^{j\omega})$.

We know that

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

Also note that

$$H^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j\omega}} = H^*\left(\frac{1}{e^{-j\omega}}\right) = H^*(e^{j\omega}).$$

Therefore,

$$|H(e^{j\omega})|^2 = H(z)H^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j\omega}}.$$

Recall that the factored form for H(z) when H(z) is a rational system function:

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}. \qquad \text{(equation 5.69)}$$

Note that for the above H(z) we can write

$$H\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^*)}{\prod_{k=1}^{N}(1-d_k z^*)}$$

and, assuming that b_0 and a_0 are real-valued,

$$H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k^* z)}.$$

Now define a new term:

$$C(z) = H(z)H^*\left(\frac{1}{z^*}\right).$$

Then we can write:

$$C(z) = H^*(z)H\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)^2\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)\left(1-d_k^* z\right)}. \qquad \text{(equation 5.72)}$$

From the above expression we see that C(z) has a zero at $z = c_k$ and also has a zero at

$$z = \left(\frac{1}{c_k^*}\right).$$

Likewise, C(z) has a pole at $z = d_k$ and also has a pole at

$$z = \left(\frac{1}{d_k^*}\right).$$

Consider a zero represented as $c_k = re^{j\theta}$ with r < 1.

Then

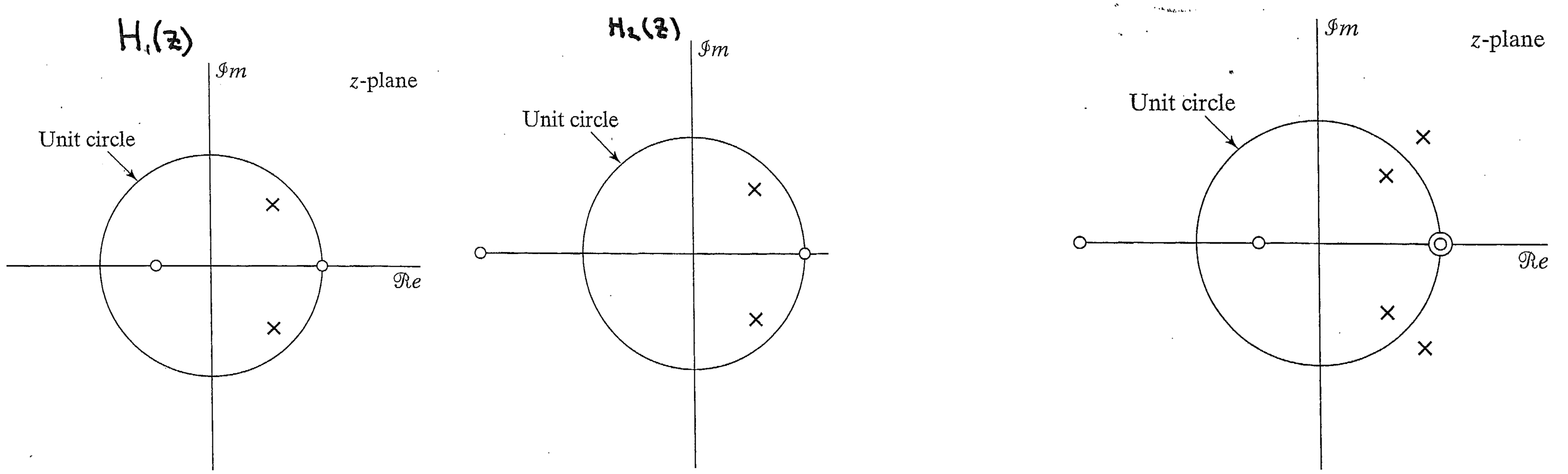
$$\frac{1}{c_k^*} = \frac{1}{re^{-j\theta}} = \frac{1}{r}e^{j\theta}.$$

Therefore, a zero of C(z) <u>inside</u> the unit circle at c_k also has a "companion" zero <u>outsid</u>e the unit circle at $\frac{1}{c_k^*}$. The same is true for poles of C(z).

For a <u>stable</u> system, the <u>poles of C(z) inside the unit circle must belong to H(z).</u> However, <u>either member of a "zero-pair" of C(z) could be assigned to H(z)</u> and the other to $H^*\left(\frac{1}{z^*}\right)$. (Both options have the same magnitude-squared frequency response function.)

Example 5.9

The following figures show the poles and zeros of two systems, $\mathbf{H_1(z)}$ and $\mathbf{H_2(z)}$, which have the same **C(z):**



(a) (b) (c) $\mathbf{C(z) = C_1(z) = C_2(z)}$

Figure 5.16 Pole-zero plots for two system functions and their common magnitude Squared function. (a) $\mathbf{H_1(z)}$, (b) $\mathbf{H_2(z)}$, and (c) $\mathbf{C(z) = C_1(z) = C_2(z)}$

<u>Example 5.10 from text</u>

The <u>C(z) for this example includes 2 pairs of complex-conjugate zeros and 1 pair of real zeros</u>. (It also includes 3 pole-pairs.) There are 4 different stable, causal H(z) functions which would have the same C(z) function, assuming that h(n) must be real-valued (and the coefficients of the implementing difference equation must be real-valued). The 4 options are shown below:

	Zeros of H(z)	Zeros of H*(1/z*)
Option 1:	z_1, z_2, z_3	z_4, z_5, z_6
Option 2:	z_1, z_2, z_6	z_4, z_5, z_3
Option 3:	z_4, z_5, z_3	z_1, z_2, z_6
Option 4:	z_4, z_5, z_6	z_1, z_2, z_3

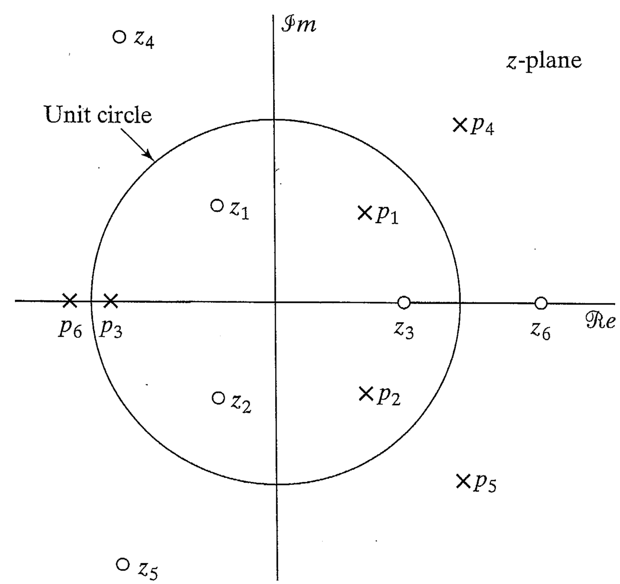


Figure 5.17 Pole-zero plot for the magnitude-squared function in Example 5.10

If h(n) is <u>not</u> constrained to be real, then there would be <u>8 possibilities for H(z) that would all have the same C(z) function</u>. There are shown below:

	<u>Zeros of H(z)</u>	<u>Zeros of H*(1/z*)</u>
Option 1:	z_1, z_2, z_3	z_4, z_5, z_6
Option 2:	z_1, z_2, z_6	z_4, z_5, z_3
Option 3:	z_4, z_5, z_3	z_1, z_2, z_6
Option 4:	z_4, z_5, z_6	z_1, z_2, z_3
Option 5:	z_1, z_5, z_3	z_4, z_2, z_6
Option 6:	z_1, z_5, z_6	z_4, z_2, z_3
Option 7:	z_4, z_2, z_3	z_1, z_5, z_6
Option 8:	z_4, z_2, z_6	z_1, z_5, z_3

<u>Note:</u> If the number of poles and zeros is not restricted, then an infinite number of versions of H(z) can have the same C(z). This is because one or more all-pass filters can be cascaded with the other poles and zeros of H(z), without changing C(z), as will be shown in the next unit.