All-Pass Filters

Consider a filter with the following system function:

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}.$$

Show that this is an all-pass filter:

$$H(e^{j\omega}) = \left.\frac{z^{-1} - a^*}{1 - az^{-1}}\right|_{z=e^{j\omega}} = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= e^{-j\omega}\left(\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}\right). \qquad \text{(equation 5.81)}$$

Since $|e^{-j\omega}| = 1$ and the other term is a ratio of complex conjugates, the overall magnitude is 1 for all ω. Notation: We will use $H_{ap}(z)$ and $H_{ap}(e^{j\omega})$ to represent an all-pass filter.

Now show that cascading an all-pass filter to another filter doesn't change the C(z) of the other filter, as shown below:

Since H(z) for a first-order all-pass filter has the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}.$$

The expression for $H_{ap}(\frac{1}{z^*})$ is

$$H_{ap}(\frac{1}{z^*}) = \frac{\left(\frac{1}{z^*}\right)^{-1} - a^*}{1 - a\left(\frac{1}{z^*}\right)^{-1}} = \frac{z^* - a^*}{1 - az^*}.$$

Taking the complex conjugate of the above gives

$$H^*_{ap}(\frac{1}{z^*}) = \frac{z - a}{1 - a^* z}.$$

Therefore,

$$H_{ap}(z)H^*_{ap}(\frac{1}{z^*}) = C(z) = \left(\frac{z^{-1} - a^*}{1 - az^{-1}}\right)\left(\frac{z - a}{1 - a^* z}\right)$$

$$= \left(\frac{1 - a^* z}{z - a}\right)\left(\frac{z - a}{1 - a^* z}\right) = 1.$$

<u>Other Properties of All-Pass Filters</u>

<u>"Companion Zero" Property</u>

If $H_{ap}(z)$ has a pole at $z = a = re^{j\theta}$,

it also has a zero at

$$z = \frac{1}{a^*} = \frac{1}{[re^{j\theta}]^*} = \frac{1}{re^{-j\theta}} = \frac{1}{r}e^{j\theta}.$$ (same angle as pole; reciprocal magnitude)

Phase of a First Order All-Pass Filter

Starting with expression (eqn.5.81) for a first-order all-pass filter:

$$H_{ap}(e^{j\omega}) = e^{-j\omega}\left(\frac{1-a^*e^{j\omega}}{1-ae^{-j\omega}}\right)$$

and representing the pole location **a** as $a = re^{j\theta}$, we can write an expression for the phase:

$$\measuredangle\, e^{-j\omega}\left(\frac{1-re^{-j\theta}e^{j\omega}}{1-re^{j\theta}e^{-j\omega}}\right) = -\omega + \measuredangle(1-re^{-j\theta}e^{j\omega}) - \measuredangle(1-re^{j\theta}e^{-j\omega})$$

$$= -\omega + \measuredangle(1-re^{j(\omega-\theta)}) - \measuredangle(1-re^{j(\theta-\omega)})$$

$$= -\omega + \measuredangle\Big(1-r\cos(\omega-\theta) - jr\sin(\omega-\theta)\Big) - \measuredangle\Big(1-r\cos(\theta-\omega) - jr\sin(\theta-\omega)\Big)$$

$$= -\omega + \measuredangle\Big(1-r\cos(\omega-\theta) - jr\sin(\omega-\theta)\Big) - \measuredangle\Big(1-r\cos(\omega-\theta) + jr\sin(\omega-\theta)\Big)$$

$$= -\omega + \arctan\left[\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] - \arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

$$= -\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]. \qquad \text{(equation 5.83)}$$

Group Delay of a First-Order All-Pass Filter

The group delay of a first order all-pass filter is defined as the negative of the derivative of the phase:

$$\text{grd}\ H_{ap}(e^{j\omega}) = -\frac{d}{d\omega}\left\{-\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]\right\}$$

$$= \frac{d}{d\omega}\left\{\omega + 2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]\right\}$$

$$= 1 + 2\;\frac{1}{1+\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)^2}\frac{d}{d\omega}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right].$$

Evaluate the derivative:

$$\frac{d}{d\omega}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] = \frac{\big(1-r\cos(\omega-\theta)\big)r\cos(\omega-\theta)-\big(r\sin(\omega-\theta)\big)r\sin(\omega-\theta)}{\big(1-r\cos(\omega-\theta)\big)^2}.$$

Therefore,

$$\text{grd } H_{ap}(e^{j\omega}) = 1 + 2\;\frac{r\cos(\omega-\theta)-r^2\cos^2(\omega-\theta)-r^2\sin^2(\omega-\theta)}{(1-r\cos(\omega-\theta))^2+r^2\sin^2(\omega-\theta)}$$

$$= 1 + 2\;\frac{r\cos(\omega-\theta)-r^2}{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)+r^2\sin^2(\omega-\theta)}$$

$$= 1 + 2\;\frac{r\cos(\omega-\theta)-r^2}{1-2r\cos(\omega-\theta)+r^2}$$

$$= \frac{1-2r\cos(\omega-\theta)+r^2+2r\cos(\omega-\theta)-2r^2}{1-2r\cos(\omega-\theta)+r^2}$$

$$= \frac{1-r^2}{1-2r\cos(\omega-\theta)+r^2} = \frac{1-r^2}{|1-re^{j\omega}e^{-j\theta}|^2} = \text{grd } H_{ap}(e^{j\omega}).$$ (see eqn. 5.85)

Note: Since r < 1 for a stable and causal system, the numerator of the above equation is always positive, as is the denominator. Therefore, the group delay of a first-order all-pass filter is always positive.

Note: The group delay for higher order all-pass filters will also be positive, since a higher order all-pass filter can be represented as a cascade of first-order all-pass filters, for which the phase and group delays add.

Since the group delay is defined as the negative of the derivative of the phase, the phase of an all-pass filter could be determined from group delay as:

$$\arg[H_{ap}(e^{j\omega})] = -\int_0^{\omega} \mathrm{grd}[H_{ap}(e^{j\phi})]\,d\phi + \arg[H_{ap}(e^{j0})] \quad 0 \le \omega \le \pi \qquad \text{(equation 5.86)}$$

Higher Order All-Pass Filters

The expression for an all-pass having an arbitrary finite number of poles and zeros is

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_k z^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1}-e_k^*)}{(1-e_k z^{-1})}\frac{(z^{-1}-e_k)}{(1-e_k^* z^{-1})}$$

where the $\{d_k\}$ are locations of the real poles and the $\{e_k\}$ and $\{e_k^*\}$ are locations of the complex poles.

From the previous expression we can evaluate $H_{ap}(e^{j0})$ for an all-pass filter having an arbitrary order as

$$H_{ap}(e^{j0}) = H_{ap}(z)\Big|_{z=1} = A\prod_{k=1}^{M_r}\frac{1-d_k}{1-d_k}\prod_{k=1}^{M_c}\frac{(1-e_k^*)}{(1-e_k)}\frac{(1-e_k)}{(1-e_k^*)} = A.$$

Therefore,

$\arg H_{ap}(e^{j0}) = \arg(A) = 0.$ (assuming that A is real and positive)

By using this result and the result of equation 5.85 (that $\text{grd}[H_{ap}(e^{j\omega})] \geq 0$)
in equation 5.86, we can obtain an important result: the phase of a stable, causal all-pass filter is non-positive for $0 \leq \omega \leq \pi$, i.e,

$\arg[H_{ap}(e^{j\omega})] \leq 0.$ $\quad 0 \leq \omega \leq \pi$ (equation 5.89)

Minimum Phase Systems (Section 5.6)

A minimum phase system is defined as a system that has all its poles and zeros inside the unit circle in the z-plane. For a given magnitude response, the minimum phase filter is one of multiple possible filters which have this common magnitude response. (As will be shown, the “minimum phase” filter actually has maximum phase among the group of filters that have the same magnitude response. The so-called minimum phase filter actually has “minimum phase delay,” as discussed later.)

Any rational H(z) can be expressed as a product of (a) a minimum phase system function that has the same magnitude response and (b) one or more all-pass filters. That is,

$H(z) = H_{min}(z)H_{ap}(z).$ (equation 5.90)

Show this:

Assume that a stable H(z) has a single zero outside the unit circle at $z = \frac{1}{c^*}$ where $c = re^{j\theta}$ and r < 1. The location of this zero can be expressed in polar form as

$$\frac{1}{re^{-j\theta}} = \frac{1}{r}e^{j\theta}.$$

H(z) can be written as

$$H(z) = H_1(z)(z^{-1} - c^*)$$

where $H_1(z)$ has all its zeros inside the unit circle.

Now multiply and divide the above expression by $(1 - cz^{-1})$:

$$H(z) = H_1(z)(1 - cz^{-1})\frac{(z^{-1} - c^*)}{(1 - cz^{-1})}.$$

The $(1 - cz^{-1})$ term in the numerator represents a zero at z = c, which is inside the unit circle since we stipulated that r < 1.

We also recognize the term $\frac{(z^{-1} - c^*)}{(1 - cz^{-1})}$ to be an all-pass filter.

Therefore, we can write

$H(z) = H_{min}(z)H_{ap}(z)$ (equation 5.93) where for this example

$H_{min}(z) = H_1(z)(1 - cz^{-1})$.

Equation 5.93 is a general result, where $H_{min}(z)$ contains the original poles of H(z), the original zeros of H(z) that are inside the unit circle, and new zeros that are conjugate reciprocals (reciprocal radius, same angle) of any original zeros of H(z) which are outside the unit circle.

$H_{ap}(z)$ consists of any original zeros of H(z) that are outside the unit circle as well as new poles at the same locations as those of the new zeros that were assigned to $H_{min}(z)$.

Properties of Minimum Phase Systems

Minimum Phase-Lag

"Minimum phase" systems actually have maximum phase. (What they have is minimum phase lag, where phase-lag is defined as the negative of the phase.) This is shown below:

Since we have seen that any rational system function H(z) can be expressed as the product of a minimum phase system $H_{min}(z)$ and an all-pass system $H_{ap}(z)$, the corresponding phase relation is

$\arg[H(e^{j\omega})] = \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$. (equation 5.101)

Consider the system version which has all its zeros inside the unit circle. (This is the minimum phase version.) Any other system having the same magnitude response can be obtained by cascading one or more all-pass filters, and we have shown that $\arg[H_{ap}(e^{j\omega})]$ is negative for $0 \le \omega \le \pi$.

Therefore, the "minimum phase" version of the system actually has the <u>maximum phase</u> function (and the <u>minimum phase-lag</u>)

<u>Minimum Group Delay Property</u>

The group delay for a rational system can be expressed as

$$grd[H(e^{j\omega})] = grd[H_{min}(e^{j\omega})] + grd[H_{ap}(e^{j\omega})].$$

We have also seen that $grd[H_{ap}(e^{j\omega})]$ is always positive.

Therefore, <u>the minimum phase version of a system has the minimum group delay</u>.

<u>Minimum Energy Delay</u>

Let $h_{min}(n)$ represent the unit sample response of the minimum phase version of a system and let h(n) represent the unit sample response of any other version of the system (which has the same magnitude frequency response).

Then

$$\sum_{n=-\infty}^{n} |h(n)|^2 \leq \sum_{n=-\infty}^{n} |h_{min}(n)|^2. \qquad \text{(equation 5.108)}$$

Consider the pole-zero plots for 4 systems which have the same magnitude response:

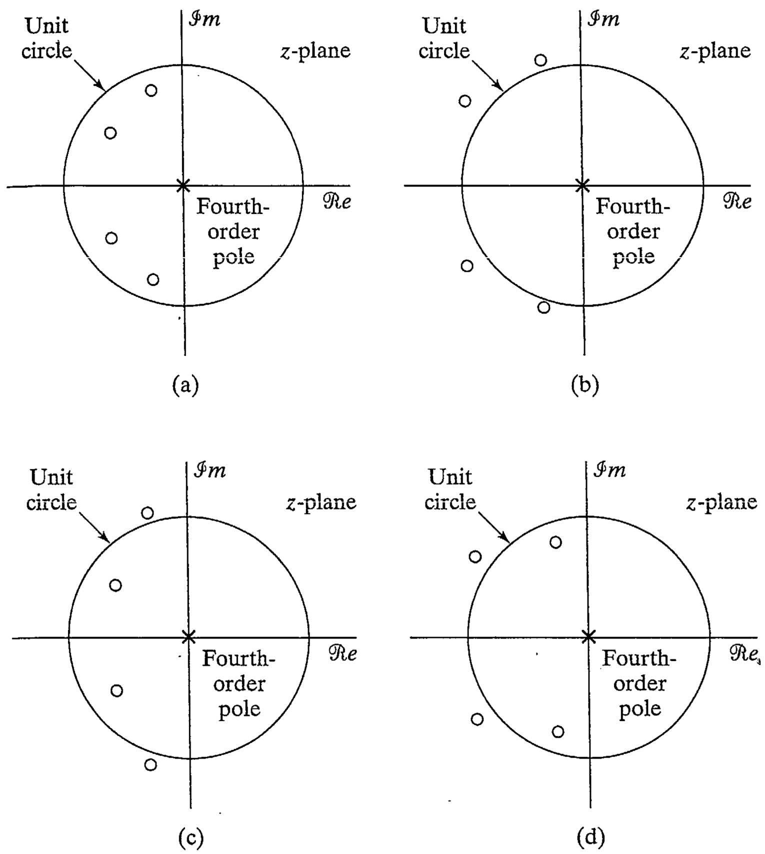

Figure 5.27 Four systems all having the same frequency response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals.

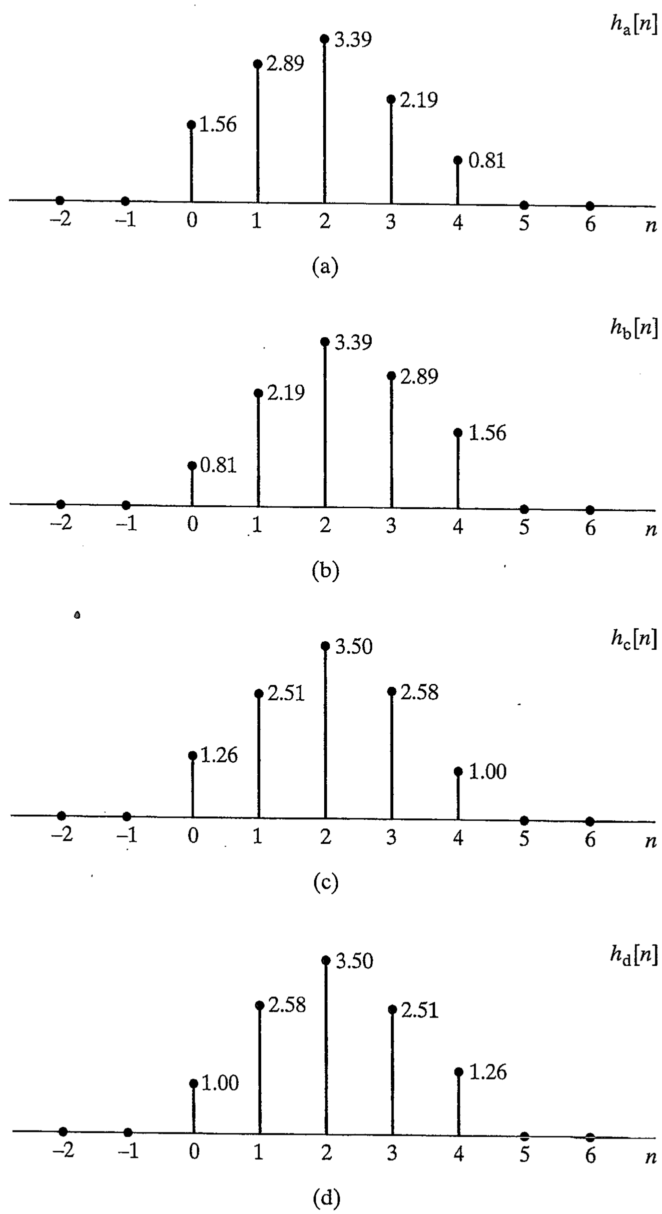

Figure 5.28 Sequences corresponding to pole-zero plots of Figure 5.27

minimum phase case

The summations in equation 5.108 are called the partial energy of h(n) and $h_{min}(n)$.

Note that due to Parseval's Theorem,

$$\sum_{n=-\infty}^{\infty} |\mathbf{h}(\mathbf{n})|^2 = \sum_{n=-\infty}^{\infty} |\mathbf{h}_{min}(\mathbf{n})|^2.$$

A plot the partial energies for the previous example is shown below:

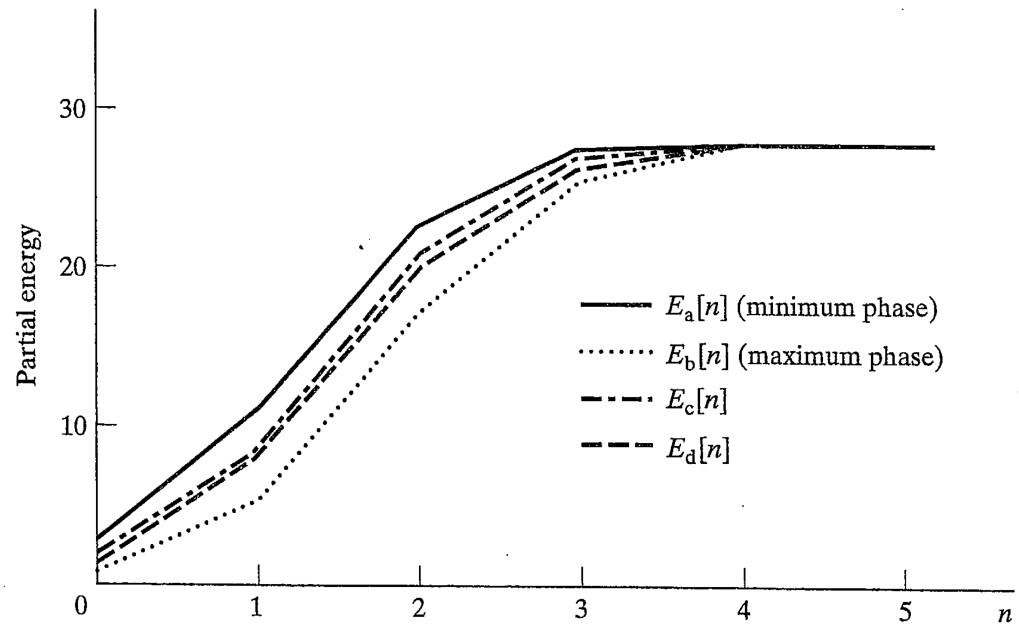

Figure 5.29 Partial energies for the four sequences of Figure 5.28. (Note that $E_a(n)$ is for the minimum phase sequence $h_a(n)$ and $E_b(n)$ is for the maximum Phase sequence $h_b(n)$.

Linear Phase

A system has linear phase if

$\measuredangle H(e^{j\omega}) = -\alpha\omega$ where α is a real constant.

For the case of linear phase

$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\alpha\omega}$

If the input to a linear time-invariant system is $x(n) = e^{j\omega_0 n}$, the output is

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega_0(n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega_0 k} = e^{j\omega_0 n} \; H(e^{j\omega})\Big|_{\omega=\omega_0}$$

If $H(e^{j\omega})$ has linear phase $= -\alpha\omega$ and has magnitude response = 1 at $\omega = \omega_0$ then

$$y(n) = e^{j\omega_0 n} \cdot 1 \cdot e^{-j\alpha\omega_0} = e^{j\omega_0(n-\alpha)}$$

$= x(n-\alpha)$, where the delay α is independent of frequency.

More generally, if $|H(e^{j\omega})| = 1$ over the entire frequency range of an input x(n), and if $\arg[H(e^{j\omega})] = -\alpha\omega$ the system output is

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = X(e^{j\omega}) \cdot 1 \cdot e^{-j\alpha\omega} = X(e^{j\omega})e^{-j\alpha\omega}.$$

Therefore, $y(n) = x(n-\alpha)$.

In the case of linear phase, the only effect of the phase response is a pure time delay, and since this delay is independent frequency, there is no phase distortion due to some frequency component of the input being delays more than others.

Note that the group delay of a system that has linear phase is

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\left\{\arg[H(e^{j\omega})]\right\} = \alpha = \text{constant}.$$

Generalized Linear Phase

If a system does not strictly satisfy the requirements for linear phase, it may still satisfy the requirements for what is called generalized linear phase. A system has generalized linear phase if its frequency response function has the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$$

where $A(e^{j\omega})$ is real, but may be negative (and therefore contribute π to the phase)

If we ignore discontinuities due to the phase contributions of $A(e^{j\omega})$, the phase is a linear function of ω:

$$-\alpha\omega + \beta$$

and the group delay is a constant, α.

Symmetry Conditions Associated with Generalize Linear Phase

As discussed above, the frequency response for a system having generalized linear can be written

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega+j\beta} \qquad \text{(equation 5.125)}$$

$$= A(e^{j\omega})\cos(\beta-\alpha\omega) + jA(e^{j\omega})\sin(\beta-\alpha\omega).$$

In general, any frequency response function $H(e^{j\omega})$ can also be written as

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h(n)\cos(\omega n) - j\sum_{n=-\infty}^{\infty} h(n)\sin(\omega n).$$

We can now use each of the above expressions for $H(e^{j\omega})$ to obtain two expressions for the tangent of the phase angle of $H(e^{j\omega})$. The first expression is

$$\tan(\text{phase angle of } H(e^{j\omega})) = \frac{A(e^{j\omega})\sin(\beta-\omega\alpha)}{A(e^{j\omega})\cos(\beta-\omega\alpha)} = \frac{\sin(\beta-\omega\alpha)}{\cos(\beta-\omega\alpha)}.$$

The other is

$$\tan(\text{phase angle of } H(e^{j\omega})) = \frac{-\sum_{n=-\infty}^{\infty} h(n)\sin(\omega n)}{\sum_{n=-\infty}^{\infty} h(n)\cos(\omega n)}.$$

Setting these equal to each other gives:

$$\frac{\sin(\beta-\omega\alpha)}{\cos(\beta-\omega\alpha)} = \frac{-\sum_{n=-\infty}^{\infty} h(n)\sin(\omega n)}{\sum_{n=-\infty}^{\infty} h(n)\cos(\omega n)}.$$

Cross-multiplying the above equation gives:

$$\sin(\beta-\omega\alpha)\sum_{n=-\infty}^{\infty} h(n)\cos(\omega n) = -\cos(\beta-\omega\alpha)\sum_{n=-\infty}^{\infty} h(n)\sin(\omega n)$$

$$\sum_{n=-\infty}^{\infty} h(n)\cos(\omega n)\sin(\beta-\omega\alpha) = -\sum_{n=-\infty}^{\infty} h(n)\sin(\omega n)\cos(\beta-\omega\alpha).$$

Now apply the trig identity: $\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b)+\sin(a-b)]$ to both sides of the above equation gives:

$$\sum_{n=-\infty}^{\infty} h(n)\frac{1}{2}\left[\sin(\beta-\omega\alpha+\omega n)+\sin(\beta-\omega\alpha-\omega n)\right] = -\sum_{n=-\infty}^{\infty} h(n)\frac{1}{2}\left[\sin(\omega n+\beta-\omega\alpha)+\sin(\omega n-\beta+\omega\alpha)\right]$$

which can also be written as:

$$\sum_{n=-\infty}^{\infty} h(n)\left[\sin[\beta+\omega(n-\alpha)]+\sin[\beta-\omega(n+\alpha)] + \sin[\beta+\omega(n-\alpha)]+\sin[-\beta+\omega(n+\alpha)]\right] = 0.$$

The second and fourth terms cancel and the first and third terms are the same, leaving

$$\sum_{n=-\infty}^{\infty} h(n)\left[\sin[(\beta+\omega(n-\alpha))]\right] = 0. \qquad \text{(equation 5.130)}$$

<u>Consider the case where $\beta = 0$ or π</u>. In this case, the above becomes:

$$\sum_{n=-\infty}^{\infty} h(n)\left[\sin[\omega(n-\alpha))]\right] = 0. \qquad \text{(equation 5.132)}$$

If we <u>also assume that α is an integer</u>, the above summation can be written:

$$\sum_{n=-\infty}^{\alpha-1} h(n)\left[\sin[\omega(n-\alpha)]\right] + 0 + \sum_{n=\alpha+1}^{\infty} h(n)\left[\sin[\omega(n-\alpha)]\right] = 0.$$

Now let $m = n - \alpha.$

When $n = \alpha - 1, \quad m = (\alpha - 1) - \alpha = -1.$

When $n = \alpha + 1, \quad m = (\alpha + 1) - \alpha = 1.$

The above summation can therefore can written as:

$$\sum_{m=-\infty}^{-1} h(m+\alpha)\sin(\omega m) + \sum_{m=1}^{\infty} h(m+\alpha)\sin(\omega m) = 0$$

$$= \sum_{m=1}^{\infty} \sin(\omega m)\left[h(m+\alpha) - h(-m+\alpha)\right] = 0.$$

The above equation will be satisfied if

$h(m+\alpha) = h(-m+\alpha)$ for all m.

Let $n = m + \alpha$. Then we can write the above condition as

$h(n) = h(2\alpha - n)$ for all n. (equation 5.131c)

$h(\alpha+1) = h(\alpha-1), h(\alpha+2) = h(\alpha-2), h(\alpha+3) = h(\alpha-3),$ etc.

Now consider the case of α = integer + (1/2). (We are still considering the case where $\beta = 0$ or π.) The condition that must be satisfied in order to have generalized linear phase is again

$$\sum_{n=-\infty}^{\infty} h(n)\left[\sin[\omega(n-\alpha))]\right] = 0. \qquad \text{(Recall equation 5.132)}$$

Now write the previous summation as

$$\sum_{n=-\infty}^{\alpha-\frac{1}{2}} h(n)\left[\sin[\omega(n-\alpha)]\right] + \sum_{n=\alpha+\frac{1}{2}}^{\infty} h(n)\left[\sin[\omega(n-\alpha)]\right] = 0.$$

Now let $m = n - \alpha - \frac{1}{2}$ in the first summation above.

When $n = \alpha - \frac{1}{2}$,

$$m = \left(\alpha - \frac{1}{2}\right) - \alpha - \frac{1}{2} = -1.$$

Now let $m = n - \alpha + \frac{1}{2}$ in the second summation above.

When $n = \alpha + \frac{1}{2}$

$$m = \left(\alpha + \frac{1}{2}\right) - \alpha + \frac{1}{2} = 1.$$

Therefore, the condition for generalized linear phase becomes

$$\sum_{m=-\infty}^{-1} h\left(m+\alpha+\frac{1}{2}\right)\sin\left(\omega(m+\frac{1}{2})\right) + \sum_{m=1}^{\infty} h\left(m+\alpha-\frac{1}{2}\right)\sin\left(\omega(m-\frac{1}{2})\right) = 0.$$

The <u>first summation</u> can be written (using a positive index of summation) as

$$\sum_{m=1}^{\infty} h\left(-m+\alpha+\frac{1}{2}\right)\sin\left(\omega(-m+\frac{1}{2})\right)$$

so the expression that must be satisfied to have generalized linear phase is

$$\sum_{m=1}^{\infty} \sin\left(\omega(m-\frac{1}{2})\right)\left\{-h\left(-m+\alpha+\frac{1}{2}\right)+h\left(m+\alpha-\frac{1}{2}\right)\right\} = 0.$$

The above condition will be satisfied if

$$h\left(m+\alpha-\frac{1}{2}\right) = h\left(-m+\alpha+\frac{1}{2}\right).$$

Now let $n = m+\alpha-\frac{1}{2}$. Then we can write the above requirement on h(n) as

$$h(n) = h\left(-n+\alpha-\frac{1}{2}+\alpha+\frac{1}{2}\right) = h(2\alpha-n). \qquad \text{(equation 5.131c)}$$

This is the same symmetry condition as we found for the case when α is an integer. Both cases are covered by the condition that 2α **= integer**.

Now consider the case where $\beta = \pi/2$ or $3\pi/2$. For either of these values of β, the general condition of equation 5.140, shown again below,

$$\sum_{n=-\infty}^{\infty} h(n)\sin[\beta + \omega(n-\alpha)] = 0$$

becomes

$$\sum_{n=-\infty}^{\infty} h(n)\cos[\omega(n-\alpha)] = 0. \qquad \text{(equation 5.134)}$$

Using an approach similar to the one used above for $\beta = 0$ or $\beta = \pi$, it can be shown that equation 5.134 is satisfied if the following symmetry condition is met:

$$h[2\alpha - n] = -h(n). \qquad \text{(equation 5.133c)}$$

Note: The symmetry conditions of equation 5.131c and equation 5.133c are sufficient conditions for generalized linear phase. However, they are not necessary conditions. For example, consider the ideal low-pass filter with linear phase:

$$H_{lp}(e^{j\omega}) = e^{-j\omega\alpha}, \quad \omega < \omega_c$$

$$= 0, \quad \omega < \omega_c < \pi.$$

The corresponding unit sample response is

$$h(n) = \frac{\sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}, \quad n \neq \alpha$$

$$= \frac{\omega_c}{\pi}, \quad n = \alpha.$$

If 2α is <u>not an integer</u>, then h(n) does not have either of the above symmetry properties, as seen in figure 5.32(c) below.

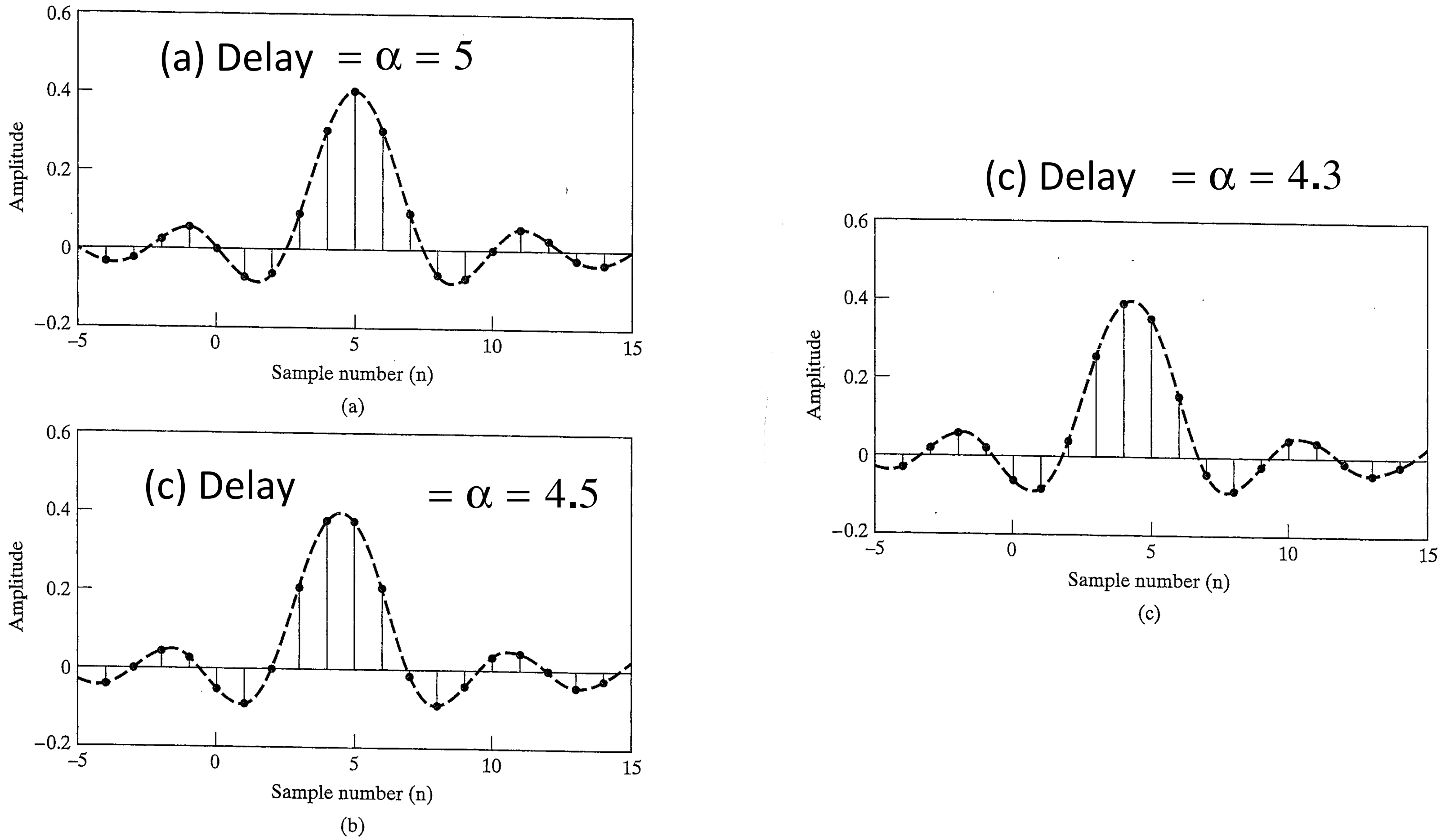

Figure 5.32 Ideal lowpass filter impulse responses, with $\omega_c = 0.4\pi$