ECE 8440 Unit 13

Section 6.9 - Effects of Round-Off Noise in Digital Filters

We have already seen that if a wide-sense stationary random signal x(n) is applied as input to a LTI system, the power density spectrum of the output y(n) is related to the power density spectrum of the input through the following relation:

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2 \, \Phi_{xx}(e^{j\omega}).$$

Assume that the input e(n) is zero-mean white noise due to round-off and the average power of this noise is σ_e^2 .

Also assume that the frequency response of that portion of the system between the entry point of the noise signal e(n) and the system output f(n) is $H_{ef}(e^{j\omega})$.

The power density spectrum of the noise in the output is therefore

$$P_{ff}(\omega) = \Phi_{ff}(e^{j\omega}) = \sigma_e^2 \, |H_{ef}(e^{j\omega})|^2 . \qquad \text{(equation 6.103)}$$

Recall that $\Phi_{ff}(e^{j\omega})$ is the Discrete Time Fourier Transform pair with the autocorrelation function $\phi_{ff}(m)$ which is defined (for the case of real signals which are wide-sense stationary) as

$$\phi_{ff}(m) = E\{(f(n)f(n+m))\}.$$

The average power of the output noise due to round-off error is

$\sigma_f^2 = E[f^2(n)]$ which is equal to $\phi_{ff}(0)$.

Therefore,

$$\sigma_f^2 = \phi_{ff}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{ff}(e^{j\omega})d\omega \qquad \text{(equation 6.104)}$$

$$= \frac{\sigma_e^2}{2\pi}\int_{-\pi}^{\pi} |H_{ef}(e^{j\omega})|^2 d\omega.$$

Applying Parseval's relation, we can also express the above as

$$\sigma_f^2 = \phi_{ff}(0) = \sigma_e^2 \sum_{k=-\infty}^{\infty} |h_{ef}(k)|^2. \qquad \text{(equation 6.105)}$$

The integral in equation 6.104 and the summation in equation 6.105 do not in general have simple solutions for higher order systems. Therefore, to obtain a more efficient way to solve for $\varphi_{ff}(0)$ we turn to z-transforms.

In order to establish background for the z-transform approach, we need to go back and generalize the development we did before in Unit 5 which led to the expression

$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega}) = |H(e^{j\omega})|^2 \, \Phi_{xx}(e^{j\omega}).$$

The first step is to generalize the previous development to permit complex-valued inputs and complex values in h(n) and x(n). As before, we assume an y(n) to be the response of an LTI system to a wide-sense stationary input.

The autocorrelation of the output process **{y(n)}** can be expressed as

$$\phi_{yy}(n,n+m) = E\left\{y^*(n)y(n+m)\right\}$$

$$E\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h^*(k)x^*(n-k)h(r)x(n+m-r)\right\}$$

$$= \sum_{k=-\infty}^{\infty} h^*(k)\sum_{r=-\infty}^{\infty} h(r)\ E\left\{x^*(n-k)x(n+m-r)\right\}.$$

Since **{x(n)}** is assumed to be wide-sense stationary,

$$E\left\{x^*(n-k)x(n+m-r)\right\} = \phi_{xx}(m+k-r).$$

Therefore, the right hand side of the original equation is independent of n, and the left-hand side must also be independent of n, that is, $\phi_{yy}(n,n+m) = \phi_{yy}(m)$.
We can therefore write

$$\phi_{yy}(m) = \sum_{k=-\infty}^{\infty} h^*(k)\sum_{r=-\infty}^{\infty} h(r)\ \phi_{xx}(m+k-r).$$

Now let $\ell = r-k$ and sum over ℓ instead of r:

$$\phi_{yy}(m) = \sum_{k=-\infty}^{\infty} h^*(k)\sum_{\ell=-\infty}^{\infty} h(\ell+k)\ \phi_{xx}(m-\ell)$$

$$= \sum_{\ell=-\infty}^{\infty} \phi_{xx}(m-\ell)\sum_{k=-\infty}^{\infty} h^*(k)\ h(\ell+k).$$

Now define

$$c_{hh}(\ell) = \sum_{k=-\infty}^{\infty} h^*(k)\ h(\ell+k)$$

which can be also be written as

$$c_{hh}(\ell) = h^*(\ell) * h(-\ell).$$

We can now write $\phi_{yy}(m)$ as

$$\phi_{yy}(m) = \sum_{\ell=-\infty}^{\infty} \phi_{xx}(m-\ell) c_{hh}(\ell)$$

which can also be represented as

$$\phi_{yy}(m) = \phi_{xx}(m) * c_{hh}(m).$$

Therefore,

$$\phi_{yy}(m) = \phi_{xx}(m) * h^*(m) * h(-m).$$

(This result is needed in developing the following material from Appendix 5, which in turn is needed as background for Section 6.9.)

Appendix A-5: Use of the z-Transform in Average Power Computations

The z-transform cannot be applied to an autocorrelation function such as $\phi_{yy}(m)$ if the mean of the signal y(n) is non-zero. To see that this is true, consider a signal

$$x(n) = x_1(n) + m_x,$$

where m_x is the mean of x(n) and $x_1(n)$ is a zero-mean signal.

The z-transform of the m_x component is

$$M_x(z) = \sum_{n=-\infty}^{-1} m_x z^{-n} + \sum_{n=0}^{\infty} m_x z^{-n}.$$

The first summation converges to $m_x\left(\frac{-z}{z-1}\right)$
with Region of Convergence of $|z| < 1$.

The second summation converges to $m_x\left(\frac{z}{z-1}\right)$
with Region of Convergence of $|z| > 1$.
Since is there no overlapping Region of Convergence for the two summations, the z-transform of the constant signal m_x does not exit.

In general, to apply the z-transform to analyze random signals, it is therefore necessary to use the autocovariance function instead of the autocorrelation function, since the mean is removed in the definition of the autocovariance function:

$$\gamma_{xx}(m) = E\{[x(n) - m_x] \bullet [x(n+m) - m_x]\}.$$

When m_x= 0, the autocovariance is equal to the autocorrelation, since in general

$$\gamma_{xx}(m) = \varphi_{xx}(m) - |m_x|^2 .$$

Based on the above discussion, consider a random signal x(n) with mean m_x. Now define a zero-mean signal as

$$x_1(n) = x(n) - m_x.$$

The autocorrelation of $x_1(n)$ is

$$\phi_{x_1x_1}(m) = E\{x_1^*(n)x_1(n+m)\}.$$

Substituting $x_1(n) = x(n) - m_x$ into the above expression gives

$$\phi_{x_1x_1}(m) = E\{\left(x(n) - m_x\right)^* \left(x(n+m) - m_x\right)\}$$

= $\gamma_{xx}(m)$,the autocovariance of x(n).

If $x_1(n)$ is the input to a LTI system, we know from previous developments in this Unit that the autocorrelation of the output $y_1(n)$ is

$$\phi_{y_1y_1}(m) = \phi_{x_1x_1}(m) * h^*(m) * h(-m).$$

We have also just shown that

$$\phi_{x_1x_1}(m) = \gamma_{xx}(m).$$

Therefore,

$$\phi_{y_1y_1}(m) = \gamma_{xx}(m) * h^*(m) * h(-m).$$

By definition, $\phi_{y_1y_1}(m)$ can also be expressed as

$$\phi_{y_1y_1}(m) = E\{y_1^*(n)y_1(n+m)\}.$$

Recall that

$$\begin{aligned} y_1(n) &= x_1(n) * h(n) \\ &= [x(n) - m_x] * h(n) \\ &= y(n) - m_x \sum_{k=-\infty}^{\infty} h(k). \end{aligned}$$

We can therefore rewrite the previous expression for $\phi_{y_1y_1}(m)$ as

$$\phi_{y_1y_1}(m) = E\{[y^*(n) - m_x^* \sum_{k=-\infty}^{\infty} h^*(k)][y(n+m) - m_x \sum_{k=-\infty}^{\infty} h(k)]\}.$$

Note that if the input to an LTI system is the random signal x(n), the expected value of the output is

$$E[y(n)] = m_y = E\{\sum_{k=-\infty}^{\infty} h(k)x(n-k)\} = m_x \sum_{k=-\infty}^{\infty} h(k).$$

Therefore, the previous expression for $\phi_{y_1y_1}(m)$ can be now expressed as

$$\phi_{y_1y_1}(m) = E\{[y^*(n) - m_y^*][y(n+m) - m_y\}.$$

which, by definition, is equal to the autocovariance of y(n). That is,

$$\phi_{y_1y_1}(m) = \gamma_{yy}(m).$$

Using this result in the previous expression of $\phi_{y_1y_1}(m) = \gamma_{xx}(m) * h^*(m) * h(-m)$

gives the important result:

$$\begin{aligned} \gamma_{yy}(m) &= \gamma_{xx}(m) * h^*(m) * h(-m) \\ &= \gamma_{xx}(m) * c_{hh}(m). \end{aligned}$$

<u>We are now prepared to use a z-transform approach for finding</u> $\gamma_{yy}(0)$.

We begin by expressing the z-transform of $c_{hh}(\ell)$:

$$Z\{c_{hh}(\ell)\} = \sum_{\ell=-\infty}^{\infty} c_{hh}(\ell) z^{-\ell}$$

$$= \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h^*(k)\, h(\ell+k) z^{-\ell}.$$

Now let $m = \ell + k$ and sum over m instead of ℓ :

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h^*(k)\, h(m) z^{-(m-k)}$$

$$= \sum_{m=-\infty}^{\infty} h(m) z^{-m} \sum_{k=-\infty}^{\infty} h^*(k)\, z^{k} = H(z)\left[\sum_{k=-\infty}^{\infty} h(k)\, (z^*)^{k}\right]^* = H(z)\left[\sum_{k=-\infty}^{\infty} h(k)\, \left(\frac{1}{z^*}\right)^{-k}\right]^*$$

$$= H(z) H^*\left(\frac{1}{z^*}\right) = C_{hh}(z).$$

Then, since $\gamma_{yy}(m) = \gamma_{xx}(m) * c_{hh}(m)$ (as already shown)

$$\Gamma_{yy}(z) = \Gamma_{xx}(z) C_{hh}(z)$$

$$= \Gamma_{xx}(z) H(z) H^*\left(\frac{1}{z^*}\right). \qquad \text{(equation A.58)}$$

What are ultimately trying to find is $\gamma_{yy}(0) = E[y^2(n)]$ for the case where the input is zero-mean white noise with

$$\gamma_{xx}(m) = \phi_{xx}(m) = \sigma_x^2\delta(m).$$

The corresponding z-transform is

$$\Gamma_{xx}(z) = \sigma_x^2$$

and therefore

$$\Gamma_{yy}(z) = \sigma_x^2 H(z) H^*\left(\frac{1}{z^*}\right).$$

The value of $\gamma_{yy}(0)$ can be found by evaluating the inverse z-transform of $\Gamma_{yy}(z)$ for the case of m=0.

Consider the case of a stable and causal system whose H(z) has the form:

$$H(z) = A\frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad \text{where } M < N. \qquad \text{(equation A.62)}$$

The Region of Convergence is $|z| > \max_k\{|d_k|\}$

where $\max_k\{|d_k|\} < 1.$

If the input is zero mean white noise with $\Gamma_{xx}(z) = \sigma_x^2$, the expression for $\Gamma_{yy}(z)$ becomes

$$\Gamma_{yy}(z) = \sigma_x^2 H(z) H^*\left(\frac{1}{z^*}\right) = \sigma_x^2 \, |A|^2 \, \frac{\prod_{m=1}^{M}(1 - c_m z^{-1})(1 - c_m^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}. \qquad \text{(equation. A.63)}$$

This can be expressed in partial fraction form as

$$\Gamma_{yy}(z) = \sigma_x^2 \left(\sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} - \frac{A_k^*}{1 - (d_k^*)^{-1} z^{-1}} \right) \qquad \text{(equation A.64)}$$

where the A_k can be found from

$$A_k = H(z) H^*\left(\frac{1}{z^*}\right)(1 - d_k z^{-1})\Big|_{z=d_k}. \qquad \text{(equation A.65)}$$

Since $\max_k \{|d_k|\} < 1$, the pole at each d_k is inside the unit circle, and the pole at each $(d_k^*)^{-1}$ is outside the unit circle. The region of convergence of $\Gamma_{yy}(z)$ is therefore

$$\max_k \{|d_k|\} < |z| < \min_k \{|(d_k^*)^{-1}|\}.$$

The <u>inverse z-transform</u> will therefore be <u>two-sided</u> and have the form:

$$\gamma_{yy}(n) = \sigma_x^2 \left[\sum_{k=1}^{N} A_k (d_k)^n u(n) + \sum_{k=1}^{N} A_k^* (d_k^*)^{-n} u(-n-1) \right].$$

We are interested in $\gamma_{yy}(0)$ which can be found as

$$\gamma_{yy}(0) = \sigma_x^2 \left(\sum_{k=1}^{N} A_k \right) = \sigma_y^2. \qquad \text{(see equation A.66)}$$

Example A.2 Noise Power Output of a 2nd Order IIR Filter

Consider a system having unit sample response

$$h(n) = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u(n). \qquad \text{(equation A.68)}$$

Using the fact, from z-transform tables, that

$$Z\{\sin(\theta n)u(n)\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}.$$

and using the z-transform property that

$$Z\{a^n x(n)\} = X(z)\Big|_{z=(z/a)},$$

the z-transform of the above h(n) can be found to be

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}.$$

To link this with previous notation, one pole is at $d_1 = re^{j\theta}$ and the other pole is at $d_2 = re^{-j\theta}$.
If the input to this system is white noise with total power $= \sigma_x^2$,
the z-transform of the autocovariance of the system output y(n) is

$$\Gamma_{yy}(z) = \sigma_x^2 H(z) H^*\left(\frac{1}{z^*}\right)$$

$$= \sigma_x^2 \left(\frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}\right)\left(\frac{1}{(1 - re^{-j\theta}z)(1 - re^{j\theta}z)}\right). \qquad \text{(equation A.69)}$$

The partial fraction representation of the above is

$$\sigma_x^2\left(\frac{A_1}{(1-re^{j\theta}z^{-1})}-\frac{A_1^*}{(1-\frac{1}{r}e^{j\theta}z^{-1})}+\frac{A_2}{(1-re^{-j\theta}z^{-1})}-\frac{A_2^*}{(1-\frac{1}{r}e^{-j\theta}z^{-1})}\right)$$

where

$$A_1=\left(\frac{1}{(1-re^{-j\theta}z^{-1})}\right)\left(\frac{1}{(1-re^{-j\theta}z)(1-re^{j\theta}z)}\right)\Bigg|_{z=re^{j\theta}}$$

and

$$A_2=\left(\frac{1}{(1-re^{j\theta}z^{-1})}\right)\left(\frac{1}{(1-re^{-j\theta}z)(1-re^{j\theta}z)}\right)_{z=re^{-j\theta}}.$$

After making the indicated substitutions and simplifying, the sum of A_1 and A_2 is

$$\left(\frac{1+r^2}{1-r^2}\right)\frac{1}{1-2r^2\cos(2\theta)+r^4}.$$

Therefore,

$$\gamma_{yy}(0)=E\{y^2(n)\}=\sigma_x^2\left(\frac{1+r^2}{1-r^2}\right)\frac{1}{1-2r^2\cos(2\theta)+r^4}. \qquad \text{(equation A.71)}$$

Note that the above expression is much more easily evaluated that the alternative methods for finding $E\{y^2(n)\}$ which are

$$E\{y^2(n)\} = \sigma_x^2 \sum_{k=-\infty}^{\infty} |h(n)|^2 = \sigma_x^2 \left| \sum_{n=0}^{\infty} \frac{r^n \sin\theta(n+1)}{\sin\theta} \right|^2$$

and

$$E\{y^2(n)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_x^2 |H(e^{j\omega})|^2 \, d\omega = \frac{\sigma_x^2}{2\pi} \int_{-\pi}^{\pi} \frac{d\omega}{|(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})|^2}.$$