ECE 8440 Unit 16

Chapter 7 - "Filter Design Techniques"

<u>Review: Methods of Designing IIR Digital Filters</u>

• Bilinear Transformation Method
- Good for transforming analog filters with flat pass-bands, including low-pass, high-pass, band-pass, and band-reject.

- Butterworth, Chebyshev, Elliptic filter types

• Impulse Invariance
- Good for transforming analog filters which have band-limited response
- Butterworth, Chebyshev, Elliptic filter types

• Minimum Mean-Squared Method
- Computer method for designing IIR filters to approximate an arbitrary frequency
 (Matlab's "yulewalk" routine can design IIR filters to approximate a piece-wise linear frequency response function)

• Other methods

Methods of Designing FIR Digital Filters

- Window Method
- Designing Optimum FIR filters Using Matlab's "firpm" algorithm

Review of Window Method

1. Find h(n) associated with desired frequency response function. (Usually get infinitely long h(n).)
2. Multiply h(n) from step 1 by finite duration window function to obtain FIR filter.

Example of Window Method

To obtain a causal FIR low-pass filter or order M, we typically start with

$$H_{lp}(e^{j\omega}) = e^{-j\omega\frac{M}{2}}, \qquad |\omega| \le \omega_c$$

$$= 0, \qquad \omega_c \le \omega \le \pi$$

Then

$$h_{lp}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lp}(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\frac{M}{2}}e^{j\omega n}\,d\omega \quad = \quad \frac{\sin[\omega_c(n-\frac{M}{2})]}{\pi(n-\frac{M}{2})}, \qquad n \ne \frac{M}{2}$$

$$= \frac{\omega_c}{\pi}, \qquad n = \frac{M}{2}$$

The unit sample response for the FIR approximation to this filter is obtained by multiplying $h_{lp}(n)$ by an appropriate window function:

$$h(n) = h_{lp}(n)w(n)$$

where $w(n)$ is 0 outside the range $0 \le n \le M$.

Since $h(n)$ is symmetric around its mid-point of $\frac{M}{2}$ and since all candidate window functions are symmetric, the resulting FIR filter with have generalized linear phase and will be either Type I or Type II, depending on whether M is even or odd.

A plot of several commonly used windows is shown below:

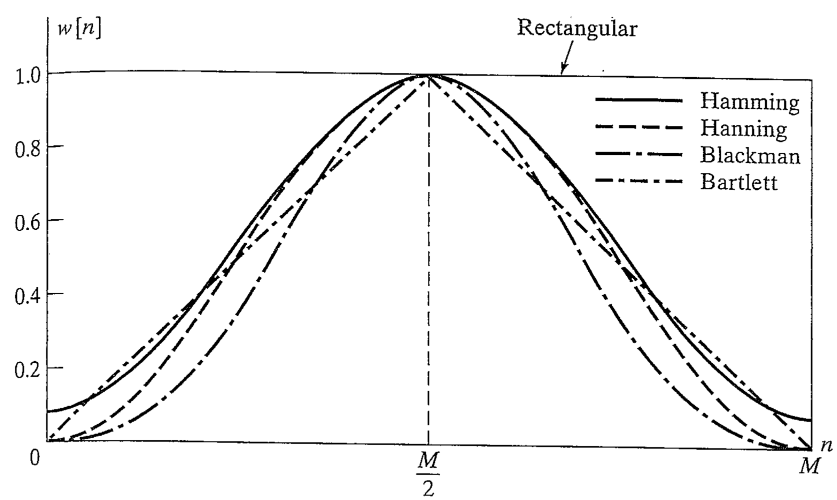


Figure 7.29 Commonly used windows.

The Fourier Transforms of the above windows are shown in the figure below:

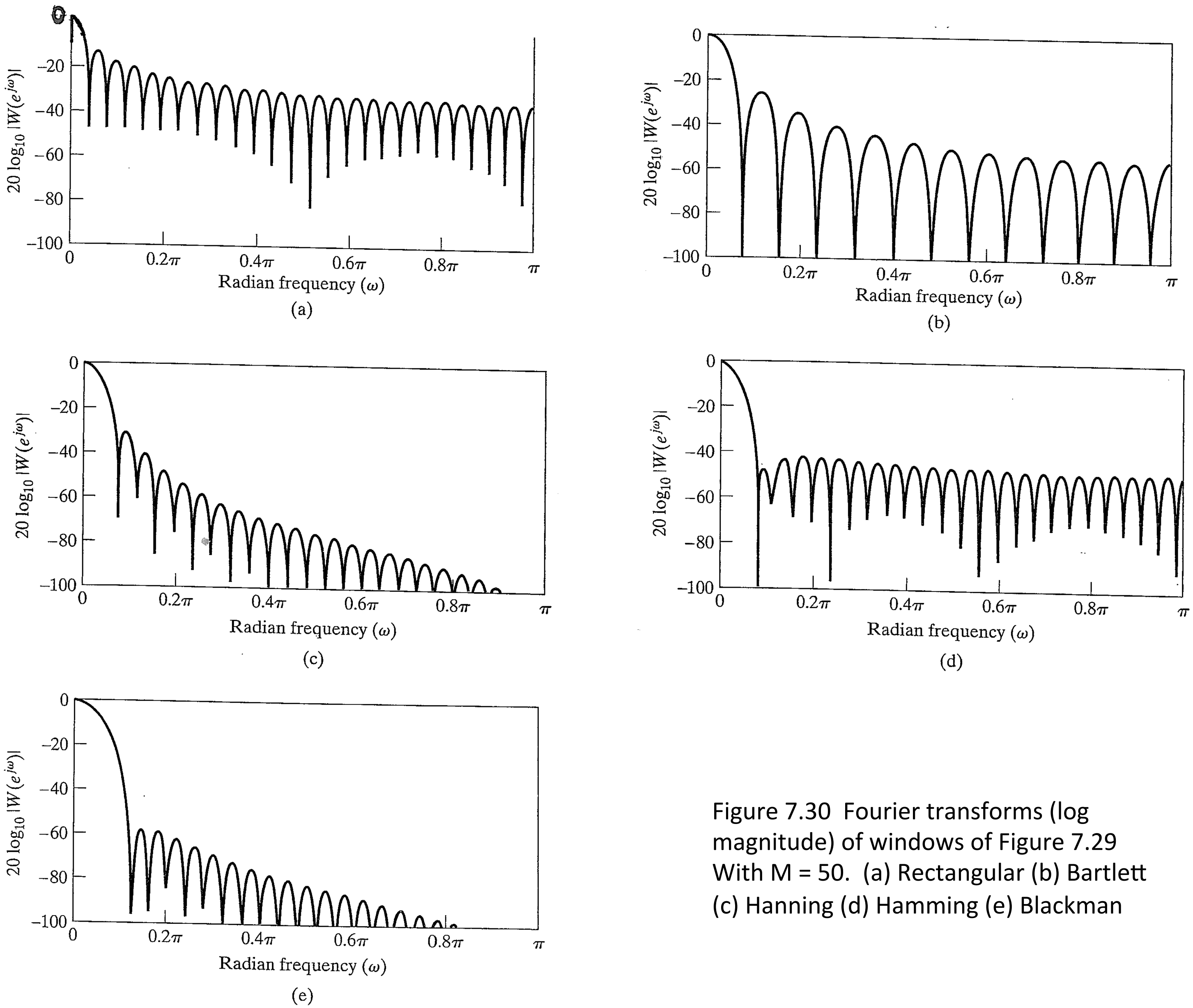


Figure 7.30 Fourier transforms (log magnitude) of windows of Figure 7.29 With M = 50. (a) Rectangular (b) Bartlett (c) Hanning (d) Hamming (e) Blackman

Frequency Domain Effects of Windowing

Multiplying $h_{lp}(n)$ by $w(n)$ corresponds to convolution of the corresponding frequency domain functions:

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lp}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

The DTFT of a typical window function and its effect in the above integral are shown below:

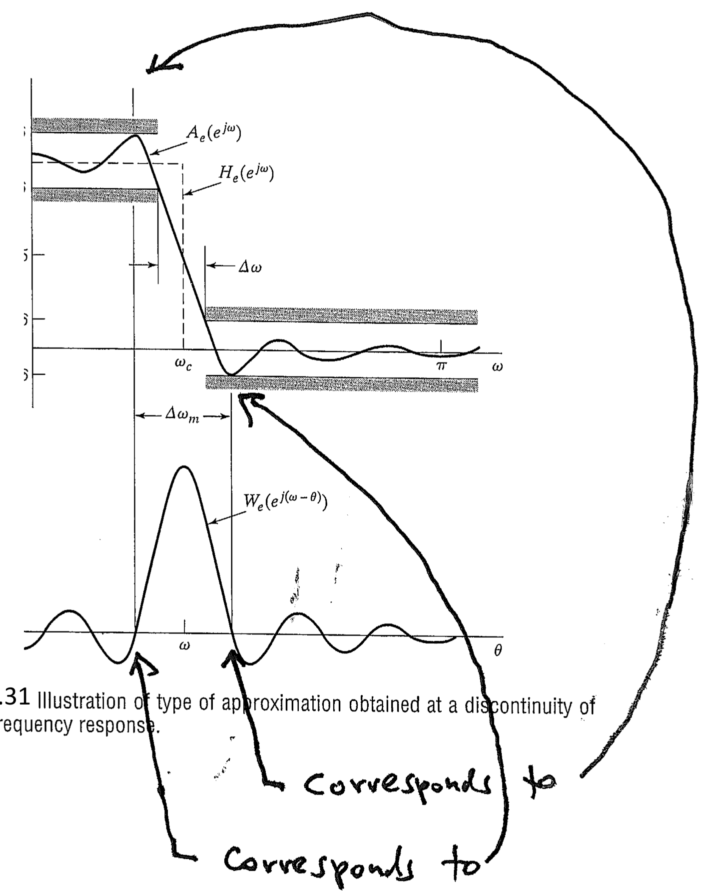
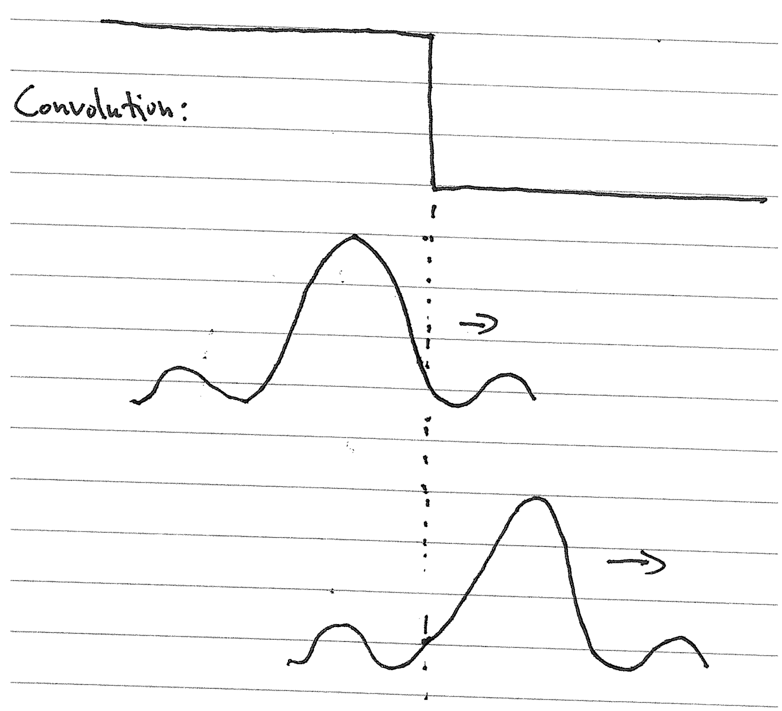


Figure 7.31 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

Related to Figure 7.31



For any window function w(n), the width of the main lobe of $W(e^{j\omega})$ can be reduced by extending the length of the window.

On the other hand, the size of the peak side-lobe of $W(e^{j\omega})$ is independent of the window length.

The following Table summarizes the features of interest in several popular window functions.

TABLE 7.2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)
Rectangular	−13	$4\pi/(M+1)$	−21
Bartlett	−25	$8\pi/M$	−25
Hanning	−31	$8\pi/M$	−44
Hamming	−41	$8\pi/M$	−53
Blackman	−57	$12\pi/M$	−74

Selection of window type and length

1. Select window type that provides sufficiently small ripple δ, using above table.
2. For selected window type, determine window length needed to provide sufficiently narrow transition region, again using above table.

High-pass filters, band-pass filters, and multi-band filters can be designed using low-pass filters as building blocks. For example, for the desired response function of the type shown below

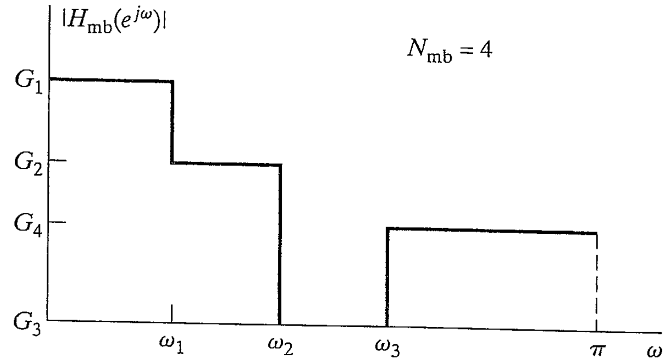


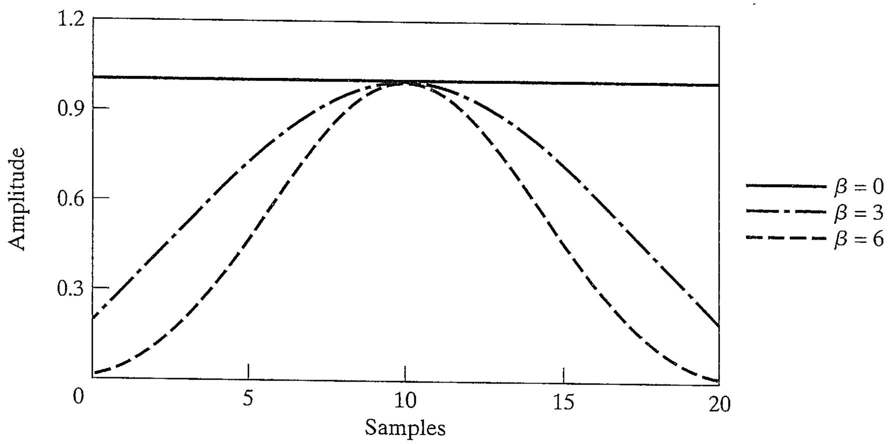
Figure 7.37 Ideal frequency response for multiband filter

We could obtain the unit sample response $h_{mb}(n)$ for the multiband filter as follows:

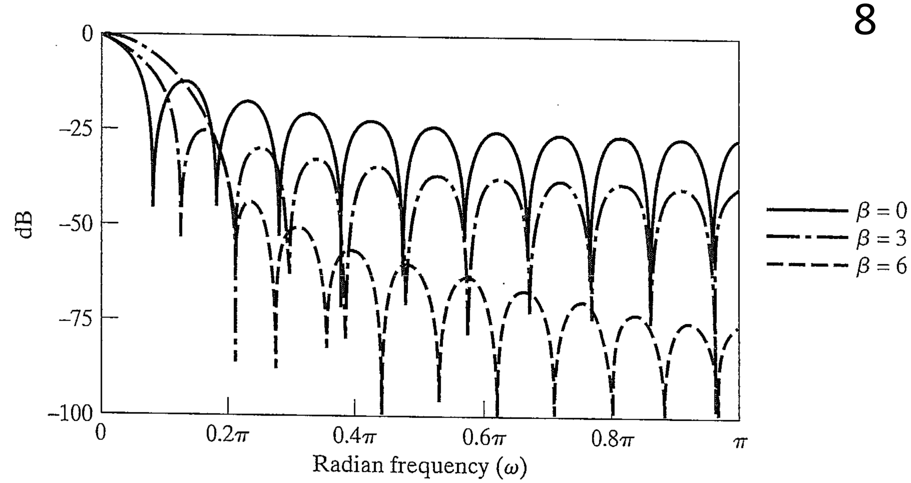
$$h_{mb}(n) = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - \frac{M}{2})}{\pi (n - \frac{M}{2})}, \qquad n \neq \frac{M}{2} \qquad \text{(equation 7.81)}$$

Kaiser Window

The "Kaiser window" is actually a family of window functions with a parameter β which can be adjusted to ensure that both design conditions (small enough ripple, narrow enough transition region) are met. (See figure below.)

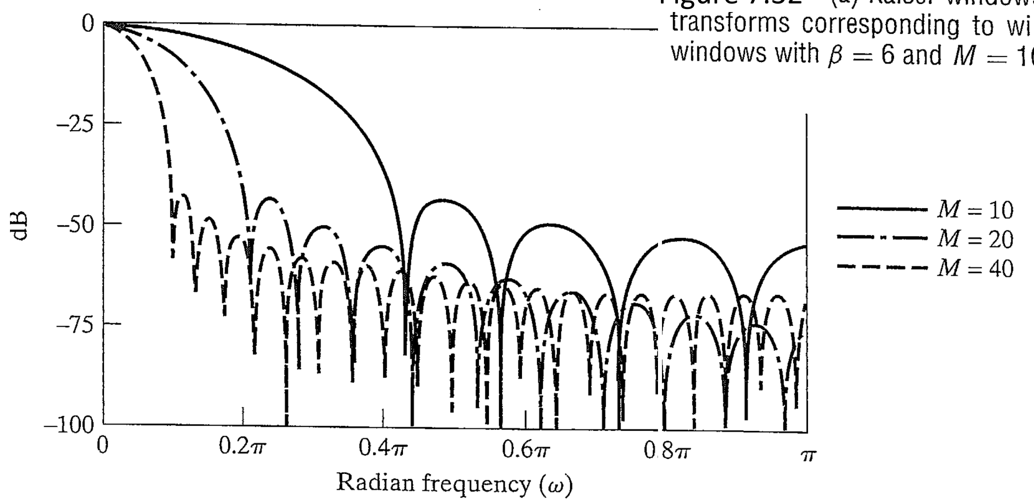


(a)



(b)

(b)



(c)

Figure 7.32 (a) Kaiser windows for $\beta = 0$, 3, and 6 and $M = 20$. (b) Fourier transforms corresponding to windows in (a). (c) Fourier transforms of Kaiser windows with $\beta = 6$ and $M = 10$, 20, and 40.

Steps in Designing FIR Filters Using the Window Method with Kaiser Windows

1. For a given value of δ , let $A = -20\log\delta$. (δ is the peak approximation error)

2. Set the parameter β as follows:

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.0, & A < 21 \end{cases}$$ (equation. 7.75)

3. Select the first approximation to the required filter length M using

$$M = \frac{A-8}{2.285\Delta\omega}$$ (equation 7.76)

The table below, which is an extended version of the table previously provided, shows the β values needed for Kaiser window in order to achieve the size of δ that is fixed for each of the other window types.

The right-hand column in the table provides the transition width for the Kaiser window for the corresponding value of δ (and therefore of A), as function of window length M.

TABLE 7.2 COMPARISON OF COMMONLY USED WINDOWS (shown again)

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Note: If one of the non-Kaiser window types is used, the right-hand column in the table also provides a better estimate of the transition width of the resulting filter, than does the approximation to the main lobe width in column three.

Optimum Approximation of FIR Filters (Section 7.7)

Start with a Type I non-causal, zero-phase FIR filter with unit sample response

$h_e(n) = h_e(-n)$ for $-L \le n \le L$

The corresponding frequency response is

$$A_e(e^{j\omega}) = \sum_{k=-L}^{L} h_e(n)e^{-j\omega n}. \qquad \text{(equation 7.88)}$$

Because of the symmetry of $h_e(n)$, $A_e(e^{j\omega})$ can be written as

$$A_e(e^{j\omega}) = h_e(0) + \sum_{n=1}^{L} 2h_e(n)\cos(\omega n).$$

which is a real, even, and periodic function of ω.

We can now make the filter causal by delaying $h_e(n)$ by L samples. The delayed $h_e(n)$ now extends from n = 0 to n = 2L. If we let M = 2L, then the h(n) for the causal filter can be represented as

$$h(n) = h_e(n - \frac{M}{2}).$$

The corresponding frequency response of the causal version of the filter is

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}. \qquad \text{(equation 7.91)}$$

Design objectives for a digital filter can be expressed in terms of a "tolerance scheme" such as the one shown below. For the low-pass filter of this figure, the parameters are ω_p , ω_s , δ_1 , δ_2 and **L**.

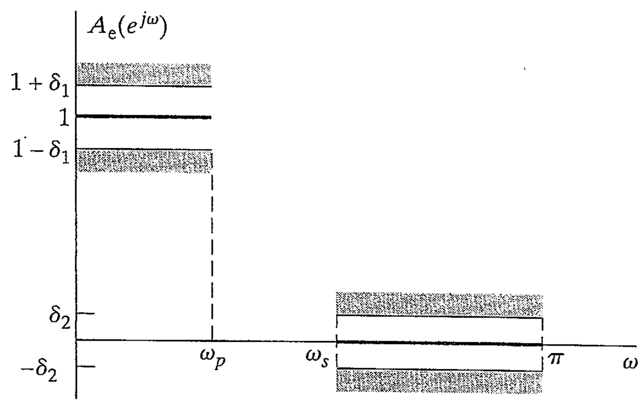


Figure 7.40 Tolerance scheme and Ideal response for lowpass filter.

<u>General approaches to find the desired filter</u>: Fix some of the parameters, then use an iterative computer solution to obtain the optimum adjustment of the others.

<u>Method of Hermann, Schussler, and Hofstetter</u>: Fix δ_1 , δ_2 , and **L**; ω_p and ω_s are allowed to vary.

<u>Method of Parks, McClellan, and Rabiner</u>: Fix ω_p, ω_s, L, and the ratio of δ_1/δ_2; δ_1 (or δ_2) are allowed to vary. This method has become the dominant approach, and is called the "Parks-McClellan algorithm." (This is the "firpm" algorithm of Matlab.)

The Parks-McClellan algorithm is based on reformulating the filter design problem as a problem of <u>polynomial approximation</u>.

It begins by expressing the $\cos(\omega n)$ terms in the expression for $A_e(e^{j\omega})$ as a sum of powers of $\cos(\omega)$ using the fact that

$$\cos(\omega n) = T_n(\cos\omega)$$

where $T_n(x)$ is the n-th order Chebyshev polynomial.

This is the same Chebyshev polynomial encountered in the design of Chebyshev IIR filters, and is defined recursively using $T_0(x) = 1$ and $T_1(x) = x$ as seeds:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) = 2x^2 - 1$$

Therefore,

$$\cos(2\omega) = 2\cos^2\omega - 1$$
$$\cos(3\omega) = 4\cos^3\omega - 3\cos\omega$$
$$\cos(4\omega) = 8\cos^4\omega - 8\cos^2\omega + 1$$

Using the above polynomial representation of $\cos(\omega n)$, we can rewrite the expression for $A_e(e^{j\omega})$ in the following form:

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos\omega)^k \qquad \text{(equation 7.93)}$$

where the a_k are related to the unit sample response of the filter.

To emphasize that we are setting up a polynomial approximation problem, note that the above equation can be written as

$$A_e(e^{j\omega}) = P(x)\big|_{x=\cos\omega}$$

where

$$P(x) = \sum_{k=0}^{L} a_k x^k.$$

The Parks-McClellan algorithm iteratively adjusts an approximation function over a disjoint sets of frequencies.

The approximation process is based on a target filter response $H_d(e^{j\omega})$ and a weighting function $W(\omega)$:

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})].$$

The target filter response $H_d(e^{j\omega})$ and a weighting function $W(\omega)$ are defined only over disjoint frequency regions. For a low-pass filter, these intervals are

$0 \le \omega \le \omega_p$ and $\omega_s \le \omega \le \pi.$

The approximation function $A_e(e^{j\omega})$ is not constrained in the transition frequency band between ω_p and ω_s (and this can cause major problems in multiband filters).

The Parks-McClellan algorithm finds the approximation function $A_e(e^{j\omega})$ which minimizes the maximum magnitude of the weighted approximation error $E(\omega)$ over all of the disjoint frequency regions included in the design specifications.

The goal of the Parks-McClellan algorithm can also be described as the following minimization problem:

$$\min_{\{h_e(n):0\le n\le L\}} \left(\max_{\omega\in F} |E(\omega)|\right).$$

For the case of a low-pass filter, F consists of the closed subsets of frequencies:
$0 \le \omega \le \omega_p$ and $\omega_s \le \omega \le \pi$.

In other words, the algorithm tries to find the $h_e(n)$ sequence that minimizes the maximum weighted approximation error over the specified frequency intervals.

If we use a weighting function of
$$W(\omega) = \begin{cases} \dfrac{1}{K}, & 0 \le \omega \le \omega_p \\ 1, & \omega_s \le \omega \le \pi \end{cases}$$

then the maximum error in the passband will be K times the maximum error in the stopband. That is, $\delta_1 = K\delta_2$.

The Parks-McClellan algorithm uses an iterative approach which is based on the following theorem:

Alternation Theorem

Let F_p denote the closed subset consisting of a disjoint union of closed subsets of the real axis x. Also let P(x) be an r-th order polynomial represented as

$$P(x) = \sum_{k=0}^{r} a_k x^k.$$

Let $D_p(x)$ denote a given desired function of x that is continuous on F_p;
Let $W_p(x)$ be a positive function which is also continuous on F_p.
Define $E_p(x)$, the weighted error, as

$$E_p(x) = W_p(x)[D_p(x) - P(x)].$$

Also define the maximum error as

$$\|E\| = \max_{x \in F_p} | E_p(x) |.$$

A necessary and sufficient condition that **P(x)** be the unique rth-order polynomial that minimizes $\|E\|$ is that $E_p(x)$ exhibit at least (r+2) alternations: i.e., there must exist at least (r+2) values $\mathbf{x_i}$ in F_p such that $\mathbf{x_1 < x_2 < \ldots < x_{r+2}}$ and such that

$$E_p(x_i) = -E_p(x_{i+1}) = \pm\|E\| \qquad \text{for } i = 1, 2, \ldots, (r+1).$$

Example: (Alternations in Polynomial Approximation) (Example 7.11)

Consider the use of a 5-th order P(x) to approximate unity and zero, respectively, over the following disjoint closed subsets:

$-1 \le x \le -0.1$ and $0.1 \le x \le 1.$

Assume that an equal weighting of 1 is assigned to each of the above subsets of x values.

According to the Alternation Theorem, the unique best approximation of a 5th order polynomial must exhibit at least 7 alternations over the closed subsets.

Based on the above information, consider whether any of the candidate polynomials (shown on the next slide) is the unique best approximation:

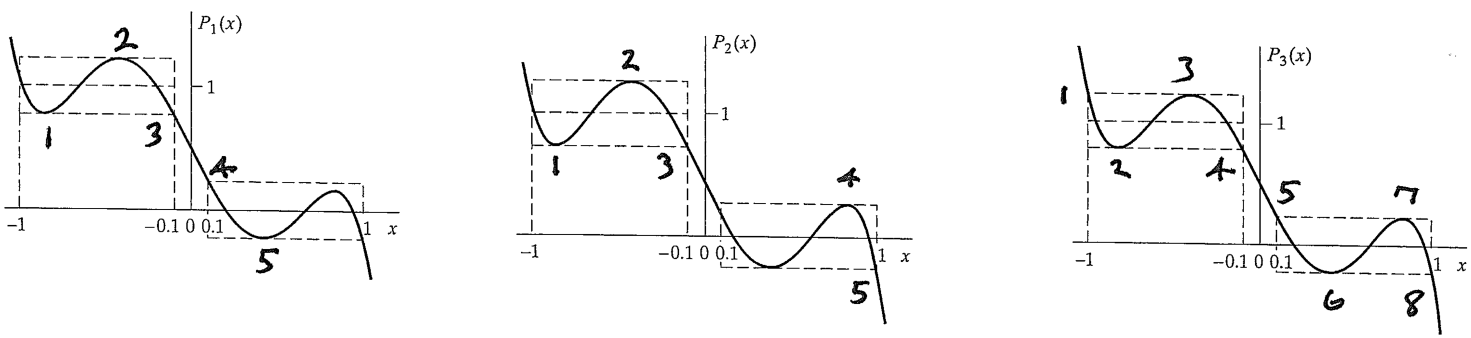


Figure 7.43 5[th]-order polynomials for Example 7.8

$P_3(x)$ has eight alternations; all points of zero slope, $x = -1$, $x = -0.1$, $x = 0.1$ and $x = 1$. Since eight alternations satisfies the alternation theorem, which specifies a minimum of seven, $P_3(x)$ is the unique optimal fifth-order polynomial approximation for this region.

From the above figure:

$P_1(x)$ has only 5 alternations.
$P_2(x)$ also has only 5 alternations.
$P_3(x)$ has 8 alternations.

Therefore, $P_3(x)$ is the unique best approximation to the two target values for the two specified closed subsets of x.

Optimal Type I Lowpass Filters

As already shown, the frequency response for a causal Type I filter has the form:

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

where $A_e(e^{j\omega})$ is a real, even, and periodic function and where M, the order of the filter, is even.

We have also seen that for a Type I filter, $A_e(e^{j\omega})$ can be written as

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos\omega)^k \quad \text{where } \mathbf{L = M/2}$$

which can also be expressed as

$$A_e(e^{j\omega}) = P(x)\Big|_{x=\cos\omega} \qquad \text{where } P(x) = \sum_{k=0}^{L} a_k x^k.$$

Note that the Alternation Theorem relates directly to $P(x)$, instead of $A_e(e^{j\omega})$.

As ω increases from **0** to π, $\cos\omega$ decreases monotonically from **1** to **-1**.

The closed subsets of ω values that we would use to specify the design requirements for a low-pass filter, and how they correspond to x values, are shown below:

$0 \le \omega \le \omega_p$ <------> $\cos\omega_p \le x \le 1$

$\omega_s \le \omega \le \pi$ <------> $-1 \le x \le \cos\omega_s$.

The frequency response for $A_e(e^{j\omega})$ for an optimum Type I lowpass filter is on the next slide, along with a plot of the corresponding polynomial function $P(x)$.

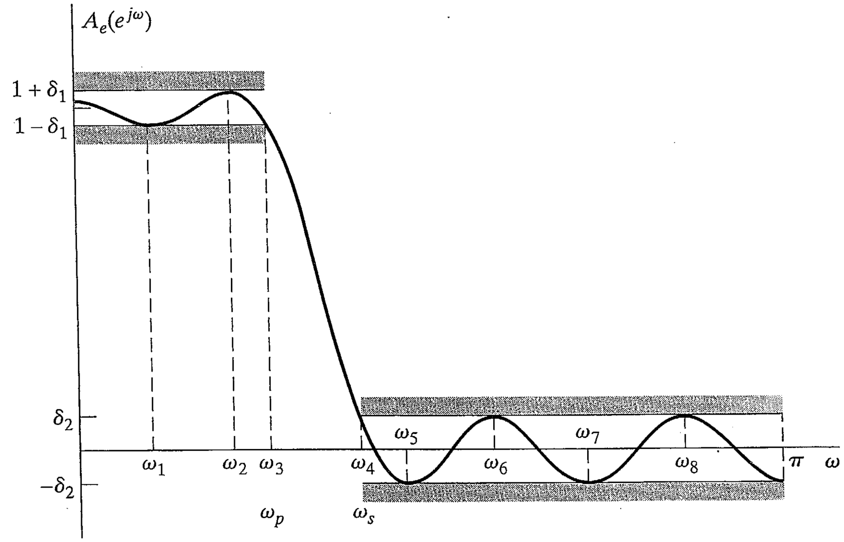


Figure 7.44 Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for $L = 7$.

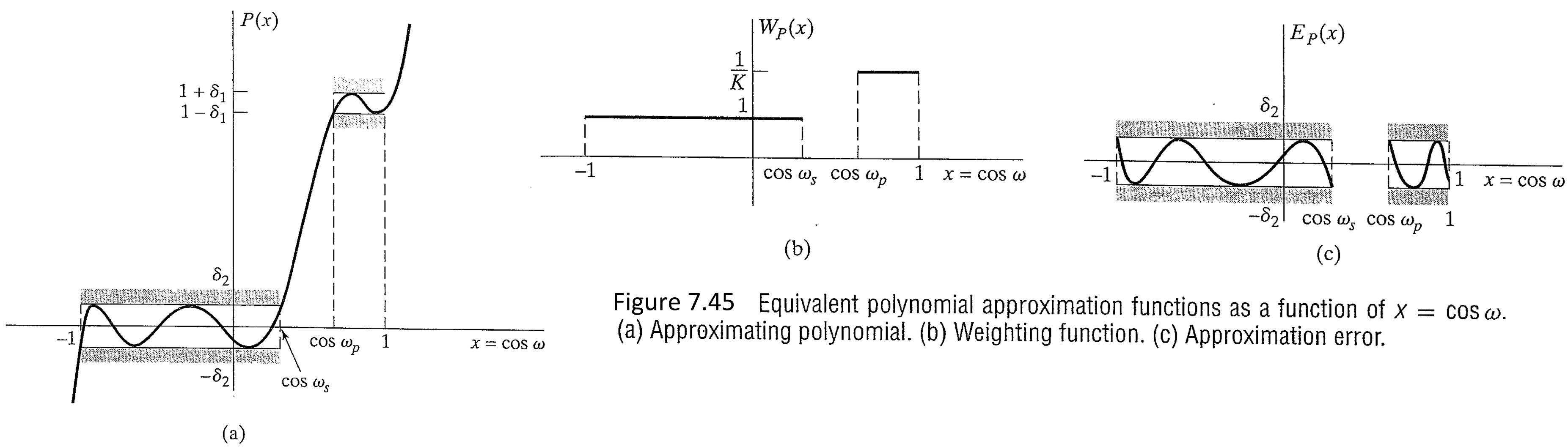


Figure 7.45 Equivalent polynomial approximation functions as a function of $x = \cos\omega$. (a) Approximating polynomial. (b) Weighting function. (c) Approximation error.

More on Alternations for FIR filters

For a low pass FIR filter having order = **M = 2L,** the maximum number of alternations, over the range $0 \le \omega \le \pi$ is **L + 3**.

Show this: Over the frequency range $0 \le \omega \le \pi$, alternations of $H(e^{j\omega})$ correspond to alterations of the L-th order polynomial $P(x)$ over $-1 \le x \le 1$, as shown in the previous figure.

An L-th order polynomial can have at most **L-1** extrema. Four additional alternations can occur at the four band edges $x = -1,\ \cos\omega_s,\ \cos\omega_p,$ and 1.

(This corresponds to $\omega = \pi, \omega_s, \omega_p, 0$). Therefore, the maximum possible number of alternations is **L +3**. A filter with L + 3 alternations is called the "extra ripple" case.

It is interesting to note that the approximating polynomial function $P(\cos\omega)$ will always have a slope of zero at $\omega = 0$ and $\omega = \pi$, even though these may not be "alternations."
We can show this by taking the derivative of $P(\cos\omega)$ as shown below:

$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

$$\frac{dP(\cos\omega)}{d\omega} = -\sin\omega\left(\sum_{k=0}^{L} k a_k (\cos\omega)^{k-1}\right) = -\sin\omega\left(\sum_{k=0}^{L-1} (k+1) a_k (\cos\omega)^{k}\right) \qquad \text{(equation 7.91)}$$

Because of the multiplier, the derivative is 0 for $\omega = 0$ and $\omega = \pi$. Note in the figure that alternations always occur at ω_p and ω_s , as discussed later in this unit.

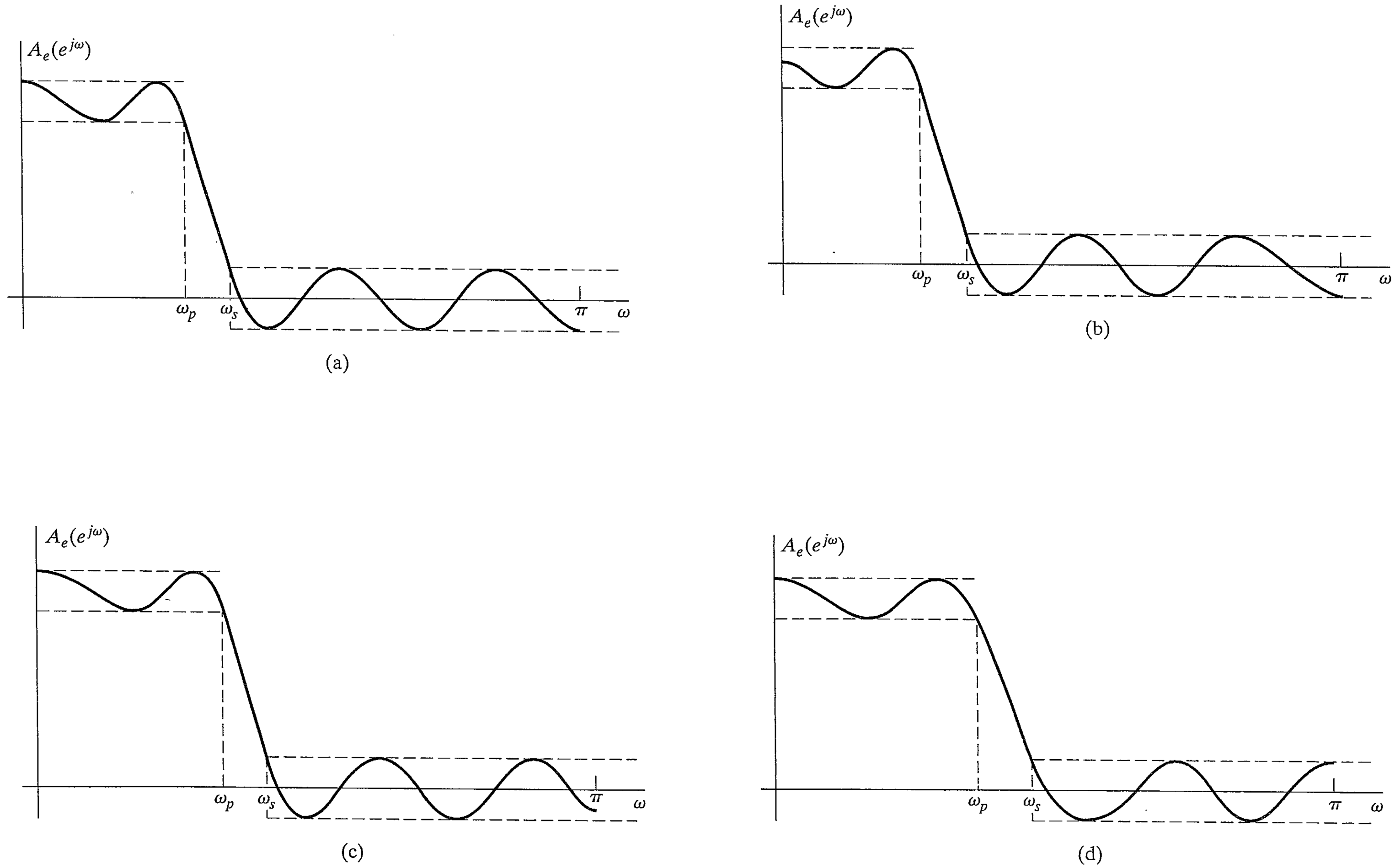


Figure 7.46 Possible optimum lowpass filter approximations for $L = 7$.
(a) $L + 3$ alternations (extraripple case).
(b) $L + 2$ alternations (extremum at $\omega = \pi$).
(c) $L + 2$ alternations (extremum at $\omega = 0$).
(d) L+2 alternati ons (extremum at $\omega = 0$ and $\omega = \pi$).

Referring to the previous figure, if there is not an alternation at ω_p, then there also cannot be an alternation at ω_s, since by definition, alternations have to represent alternating "error-high" condition and "error-low" condition.

If neither ω_p nor ω_s is an alternation, then there are not enough other ways to have the **L +2** needed for an optimum filters. Therefore, the optimum low-pass filter must have alternations at both ω_p and ω_s.

<u>Optimum Type II Low Pass Filters</u>

The unit sample response for a causal Type II filter whose h(n) extends over $0 \le n \le M$ satisfies the following symmetry condition, where M is odd:

$$h(n) = h(M - n).$$

Therefore, the frequency response can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{M} h(k)e^{-j\omega n} = \sum_{n=0}^{\frac{M-1}{2}} h(n)[e^{-j\omega n} + e^{-j\omega(M-n)}]$$

$$= e^{-j\omega M/2} \sum_{n=0}^{\frac{M-1}{2}} h(n)[e^{-j\omega n}e^{j\omega M/2} + e^{j\omega M/2}e^{-j\omega(M-n)}]$$

$$= e^{-j\omega M/2} \sum_{n=0}^{\frac{M-1}{2}} h(n)[e^{j\omega(\frac{M}{2}-n)} + e^{-j\omega(\frac{M}{2}-n)}]$$

$$= e^{-j\omega M/2} \sum_{n=0}^{\frac{M-1}{2}} 2h(n)\cos\left(\omega\left(\frac{M}{2} - n\right)\right) \qquad \text{(equation 7.106)}$$

Now let $k = \frac{M+1}{2} - n.$

When $n = 0, \;\; k = \frac{M+1}{2}.$

When $n = \frac{M-1}{2}, \quad k = \frac{M+1}{2} - \frac{M-1}{2} = 1.$

We can now write the above summation for the frequency response as

$$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=1}^{\frac{M+1}{2}} 2h\left(\frac{M+1}{2} - k\right)\cos\left(\omega\left(\frac{M}{2} - \left(\frac{M+1}{2} - k\right)\right)\right)$$

$$= e^{-j\omega M/2} \sum_{k=1}^{\frac{M+1}{2}} 2h\left(\frac{M+1}{2} - k\right)\cos\left(\omega\left(k - \frac{1}{2}\right)\right).$$

If we also define b(k) as

$$b(k) = 2h(\frac{M+1}{2} - k)$$

the above expression for the frequency response can be written as

$$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=1}^{\frac{M+1}{2}} b(k)\cos\left(\omega\left(k - \frac{1}{2}\right)\right). \qquad \text{(equation 7.107)}$$

Using the trig identity: $\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
and a lot of manipulation, the above can be written as

$$H(e^{j\omega}) = e^{-j\omega M/2}\cos(\omega/2)\sum_{k=0}^{\frac{M-1}{2}}\tilde{b}(k)\cos\left(\omega k\right).$$

As we have already seen, the above summation can be expressed as a trigonometric polynomial $P(\cos\omega)$ so that $H(e^{j\omega})$ can be written in the form:

$$H(e^{j\omega}) = e^{-j\omega M/2}\cos(\omega/2)P(\cos\omega) \qquad \text{(equation 7.109a)}$$

where $P(\cos\omega) = \sum_{k=0}^{L} a_k(\cos\omega)^k$. (equation 7.109b)

<u>Therefore, in setting up a target response for</u> $P(\cos\omega)$, <u>we must take into account the</u> <u>built-in</u> $\cos(\omega/2)$ <u>term for Type II filters</u>.

<u>For a lowpass filter</u>, the target for should be

$$H_d(e^{j\omega}) = D_P(\cos\omega) = \frac{1}{\cos(\omega/2)}, \quad \text{for } 0 \le \omega \le \omega_p \qquad \text{(equation 7.110)}$$

$$= 0, \qquad \text{for } \omega_s \le \omega \le \pi.$$

If we want to specify a ratio K of passband to stopband ripple, the weighting function for Type II filters should also take into account the built in $\cos(\omega/2)$ term, and should be

$$W(\omega) = W_P(\cos\omega) = \frac{\cos(\omega/2)}{K}, \qquad \text{for } 0 \le \omega \le \omega_p$$

$$= \cos(\omega/2), \qquad \text{for } \omega_s \le \omega \le \pi \qquad \text{(equation 7.111)}$$

$$W(\omega) = W_p(\cos\omega) = \frac{\cos(\omega/2)}{K}, \qquad \text{for } 0 \le \omega \le \omega_p$$ (repeat of equation 7.111)

$$= \cos(\omega/2), \qquad \text{for } \omega_s \le \omega \le \pi$$

To see this, we can express the term to be minimized in the Alternation Theorem, using the above expressions for $D_P(\cos\omega)$ and $W_P(\cos\omega)$ for Type II filters, as follows:

$$E_P(\cos\omega) = W_P(\cos\omega)[D_P(\cos\omega) - P(\cos\omega)]$$

$$= \frac{\cos(\omega/2)}{K}\left[\frac{1}{\cos(\omega/2)} - P(\cos\omega)\right], \qquad \text{for } 0 \le \omega \le \omega_p$$

$$= \frac{1}{K}\left[1 - \cos(\omega/2)P(\cos\omega)\right], \qquad \text{for } 0 \le \omega \le \omega_p.$$

For $\omega_s \le \omega \le \pi$, the corresponding expressions for $E_P(\cos\omega)$ is

$$E_P(\cos\omega) = \cos(\omega/2)\left[0 - P(\cos\omega)\right] = -\cos(\omega/2)P(\cos\omega).$$

For Type III and Type IV filters, we must also take into account built-in frequency dependent functions when specifying a target response for , as summarized below:

<u>Type III filters</u>

$$H(e^{j\omega}) = e^{-j\omega M/2}\sin(\omega)\sum_{k=0}^{\frac{M}{2}-1} a_k(\cos\omega)^k$$

<u>Type IV filters</u>

$$H(e^{j\omega}) = e^{-j\omega M/2}\sin(\omega/2)\sum_{k=0}^{\frac{M-1}{2}} a_k(\cos\omega)^k$$