Parks-McClellan Algorithm

The Parks-McClellan Algorithm is a computer method to find the unit sample response h(n) for an optimum FIR filter that satisfies the conditions of the Alternation Theorem.

The method used by the Parks-McClellan Algorithm is summarized below:

Let $H_d(e^{j\omega})$ denote the desired frequency response over the specified disjoint frequency intervals. Also let $A_e(e^{j\omega})$ denote the frequency response for the optimum approximation.

Then, because of the Alternation Theorem, we know $A_e(e^{j\omega})$ will satisfy the following set of equations:

$$W(\omega_i)[H_d(e^{j\omega_i}) - A_e(e^{j\omega_i})] = (-1)^{i+1}\delta, \quad i = 1,2,\ldots,(L+2). \qquad \text{(equation 7.112)}$$

Dividing both sides by $W(\omega_i)$ gives

$$[H_d(e^{j\omega_i}) - A_e(e^{j\omega_i})] = \frac{1}{W(\omega_i)}(-1)^{i+1}\delta, \quad i = 1,2,\ldots,(L+2)$$

which can also be expressed as

$$H_d(e^{j\omega_i}) = A_e(e^{j\omega_i}) + \frac{1}{W(\omega_i)}(-1)^{i+1}\delta\ , \qquad i = 1,2,\ldots,(L+2).$$

In matrix form, this can be expressed as

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & \dfrac{1}{W(\omega_1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & \dfrac{-1}{W(\omega_2)} \\ \vdots & & & & & \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & \dfrac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_L \\ \delta \end{bmatrix} = \begin{bmatrix} H_d(e^{j\omega_1}) \\ H_d(e^{j\omega_2}) \\ \vdots \\ H_d(e^{j\omega_{L+2}}) \end{bmatrix}$$

where $x_i = \cos\omega_i$.

The frequencies ω_i, $i = 1, 2, \cdots, L+2$, are the frequencies where alternations occur.

Based on the above set-up, the Parks-McClellan Algorithm uses the following steps to find the optimum filter:

Step 1. Guess the values of the frequencies, ω_i, $i = 1, 2, \cdots, L+2$, where the alternations will occur. (The frequencies ω_p and ω_s are fixed, and must be included as part of the above set.)

Step 2. Solve for δ, the approximation error at the "guessed" alternation frequencies ω_i using

$$\delta = \frac{\sum_{k=1}^{L+2} b_k H_d(e^{j\omega_k})}{\sum_{k=1}^{L+2} \frac{b_k (-1)^{k+1}}{W(\omega_k)}} \qquad \text{(equation 7.114)}$$

where

$$b_k = \prod_{\substack{i=1 \\ i\neq k}}^{L+2} \frac{1}{(x_k - x_i)} \qquad \text{(equation 7.115)}$$

where again $x_i = \cos\omega_i$.

Now assume that $W(\omega_k) = 1/K$ for all ω_i in the passband $(0 \le \omega_i \le \omega_p)$ and that $W(\omega_k) = 1$ for all ω_i in the stopband ($\omega_s \le \omega_i \le \pi$).

At the current values of the alternation frequencies ω_i the current version of the filter $A_e(e^{j\omega})$ will then satisfy the following:

$$A_e(e^{j\omega_i}) = 1 \pm K\delta \quad \text{for} \quad 0 \le \omega_i \le \omega_p$$

and

$$A_e(e^{j\omega_i}) = \pm\delta \quad \text{for} \quad \omega_s \le \omega_i \le \pi.$$

<u>Step 3</u>. Use the Lagrange interpolation formula to calculate the value of $A_e(e^{j\omega})$ over a fine-grain of frequencies between the initial set of ω_i values, using

$$A_e(e^{j\omega}) = P(\cos\omega) = \frac{\sum_{K=1}^{L+1}[d_k/(x-x_k)]C_k}{\sum_{K=1}^{L+1}[d_k/(x-x_k)]} \qquad \text{(equation 7.116a)}$$

where $x = \cos\omega$ and $x_i = \cos\omega_i$ and

$$C_k = H_d(e^{j\omega}) - \frac{(-1)^{k+1}\delta}{W(\omega_k)} \qquad \text{(equation 7.116b)}$$

and $$d_k = \prod_{\substack{i=1 \\ i\neq k}}^{L+1} \frac{1}{(x_k - x_i)} = b_k(x_k - x_{L+2}). \qquad \text{(equation 7.116c)}$$

If, in the calculation of $A_e(e^{j\omega})$ over a dense set of frequencies, it is found that the weighted approximation error $E(\omega)$, which is defined as

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})]$$

satisfies $|E(\omega)| \leq |\delta|$ at all frequencies in the specified disjoint frequency intervals, then the optimum filter has been found and the design process stops. Otherwise, the following step is implemented:

<u>Step 4</u>. Repeat the above process, starting at step 2, but this time using new "guesses" of the alternation frequencies: This time set the guesses equal to the frequencies where the previous $A_e(e^{j\omega})$ had the largest L+2 error peaks, as determined in step 3. (As before, ω_p and ω_s must be included in this set.)

The following figure demonstrates Steps 3 and 4 at an intermediate cycle of the above process:

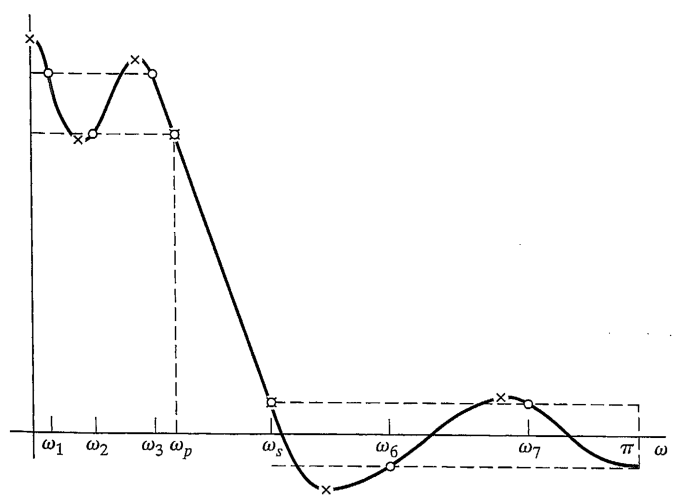


Figure 7.49 Illustration of the Parks-McClellan algorithm for equiripple approximation.

The above steps are repeated until the extremal points ω_i do not change by more than some small prescribed amount from the previous iteration.

The following flow chart gives another view of the iterative process used to find the optimum filter:

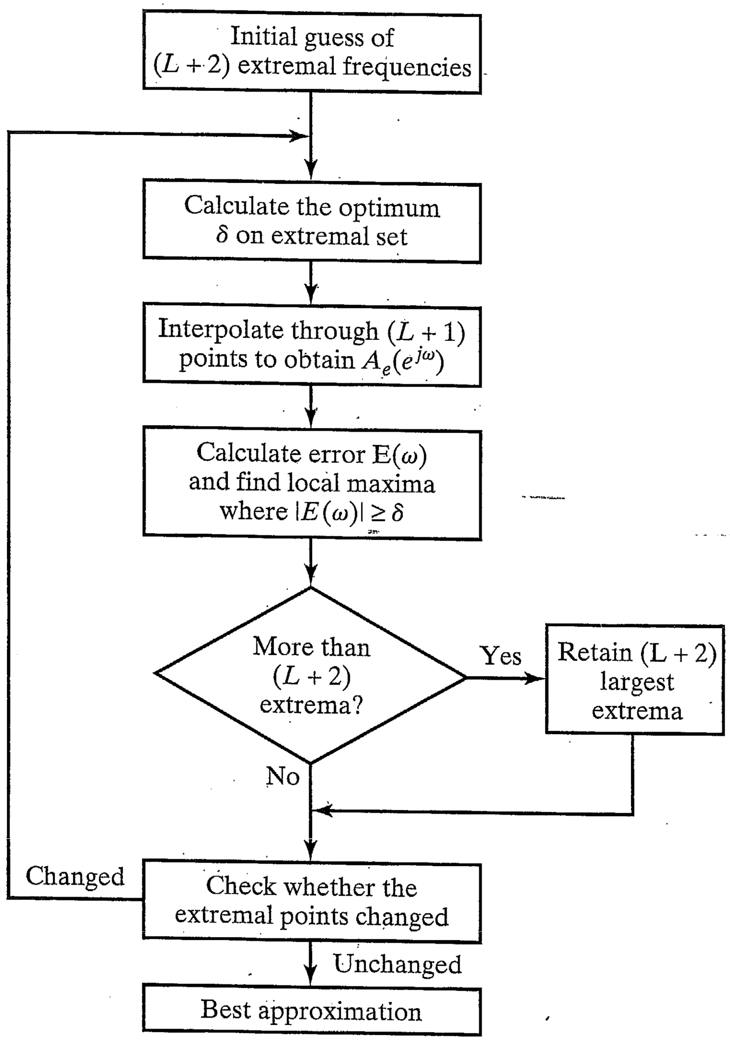


Figure 7.50 Flowchart of Parks-McClellan algorithm

After the above process has converged, the values of h(n), which are also the coefficients of the resulting filter, can be found as follows:

Step 1. Evaluate $H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$ at R equally spaced samples:

$$\omega_k = k\frac{2\pi}{R}, \qquad 0 \le k \le R-1$$

where $R \ge M$, using the interpolation formula of step 3 above.

Step 2. Take the inverse DFT of the R samples of step 1. The first M outputs of the IDFT are the desired h(n) for $0 \le n \le M$.

(See the figure below from Chapter 8 that relates to how we obtain the final values for h(n).)

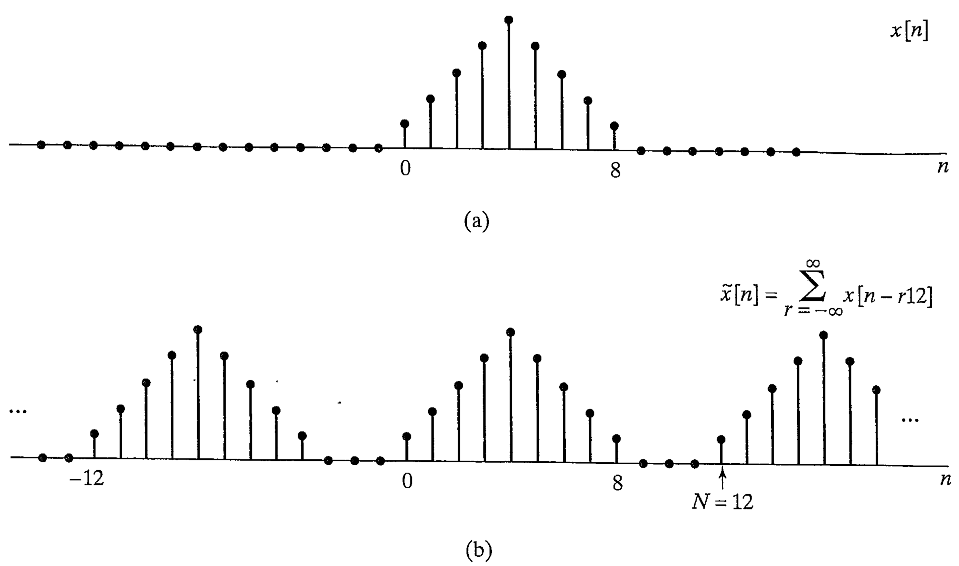


Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

Characteristics of Optimum FIR Filters

The Parks-McClellan Algorithm finds the optimum filter (the one that minimizes the maximum weighted approximation error) where the values of ω_s , ω_p , and **M** (the filter order) are fixed.

It is interesting to note that the size of the resulting approximation error varies with ω_p for the case where the transition width and the error weighting function are fixed, as shown in the figure below:

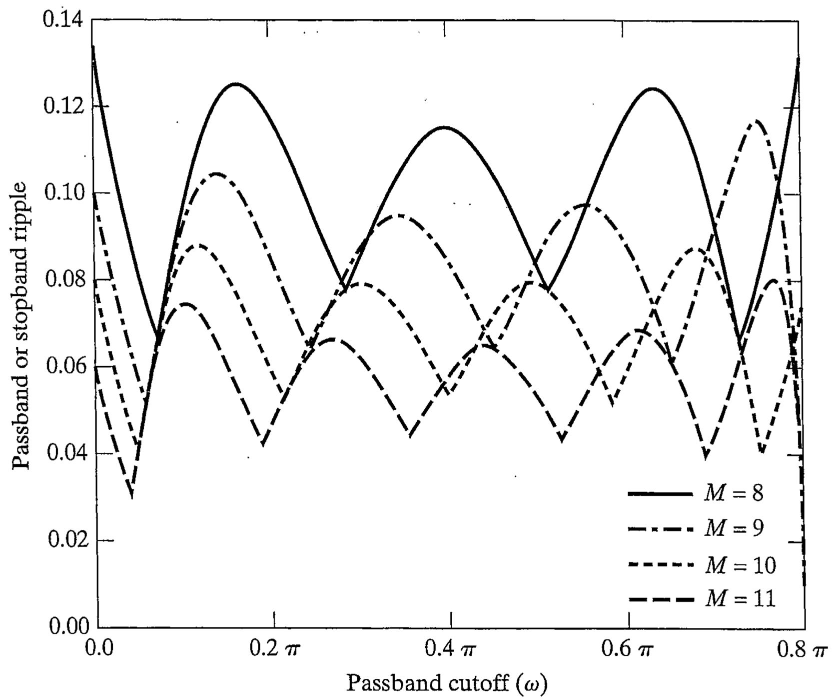


Figure 7.51 Illustration of the dependence of passband and stopband error on cutoff frequency for optimal approximations of a lowpass filter. For this example, $K = 1$ and $(\omega_s - \omega_p) = 0.2\pi$.

The cases where local minima occur in the above figure correspond to "extra ripple" cases, which have **L+3** alternations instead of **L+2.**

It is also interesting to note from the figure that increasing the filter order (e.g., from M = 9 to M = 10) may not reduce the approximation error, for some sets of design parameters.

The reason this can happen is that even-order filters are Type I filters while odd-order filters are Type II filters, which are fundamentally different.

However, the performance of any Type I filters can be improved, for any set of parameters, by increasing its order by 2 (to the next available order for Type I).

The same is true for Type II filters.

<u>For optimum FIR filters</u>, it has been determined that the order required to meet a set of design requirements can be approximated by

$$M = \frac{-10\log_{10}\delta_1\delta_2 - 13}{2.324\Delta\omega} \quad \text{where } \Delta\omega = \omega_s - \omega_p. \qquad \text{(equation 7.117)}$$

If $\delta_1 = \delta_2 = \delta$, this estimate for M becomes

$$M = \frac{-20\log_{10}\delta - 13}{2.324\Delta\omega}.$$

In order to <u>compare performance with a filter designed using the Kaiser window</u>, let $A_0 = -20\log_{10}\delta$

Then the estimated value of M for the optimum filter becomes:

$$M_0 = \frac{A_0 - 13}{2.324\Delta\omega}.$$

Expressing A_0 in terms of the filter order gives:

$$A_0 = 2.324(\Delta\omega)M_0 + 13 \text{ db.}$$

Recall for the Kaiser window method, the estimate for the required filter order was

$$M_K = \frac{A_K - 8}{2.285\Delta\omega}$$

so that for the Kaiser window method: .

$$A_K = 2.285(\Delta\omega)M_K + 8 \text{ db.}$$

If $M_0 = M_K$, then $A_0 \approx A_K + 5$.

Optimum Bandpass Filters

- Band-pass filters can have > L + 3 alternations
- In band-pass filters, local extrema can occur in transition regions.

Example

The desired frequency response is

$$H_d(e^{j\omega}) = \begin{cases} 0, & 0 \le \omega \le .3\pi \\ 1, & .35\pi \le \omega \le .6\pi \\ 0, & .7\pi \le \omega \le \pi \end{cases} \qquad \text{(equation 7.124)}$$

with the following weighting function:

$$W(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega \le .3\pi \\ 1, & .35\pi \le \omega \le .6\pi \\ .2, & .7\pi \le \omega \le \pi \end{cases}$$

Therefore $\delta_1 = \delta_2$, and $\delta_3 = 5\delta_1$.

If we select the filter order as M = 74, the corresponding value of L is L = (M/2) = 37.

According to the Alternation Theorem, the optimum filter must have at least L + 2 = 39 alternations.

The filter whose frequency response is shown in the figure below has 39 alternations and is therefore optimum; however, this filter would be unacceptable due to the non-monotonic response in the transition region.

(The kind of characteristic is not ruled out by the statement of the Alternation Theorem.)

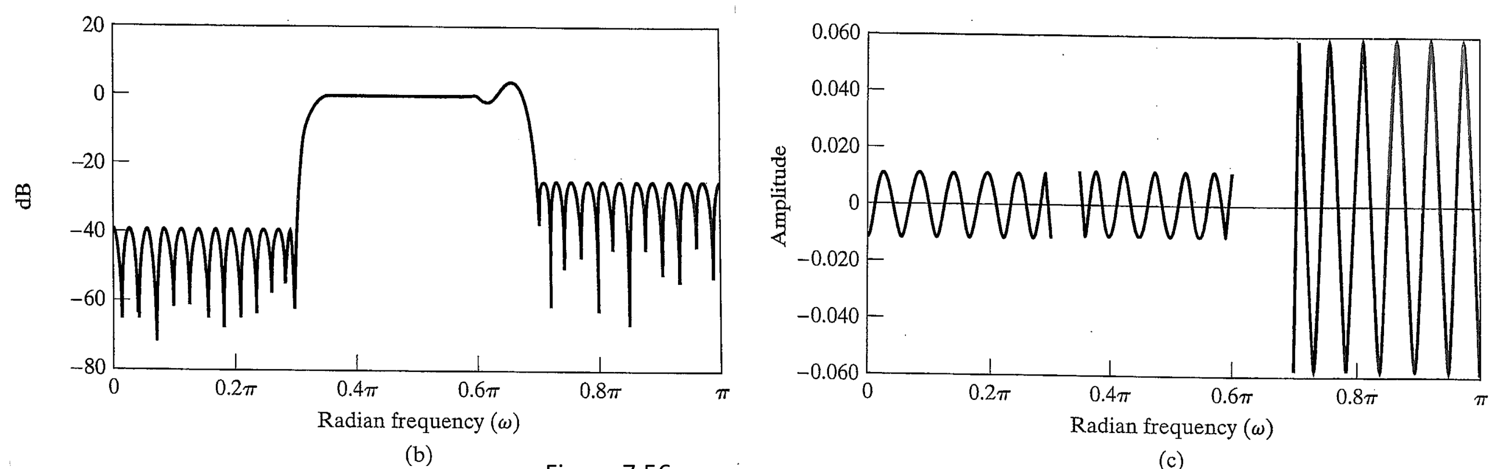


Figure 7.56 Optimum FIR bandpass filter for $M = 74$. (a) Impulse response. (b) Log magnitude of the frequency response. (c) Approximation error (unweighted).

Example: (Compensation for Zero-Order Hold)

Recall from Chapter 4 the structure for a system which has an continuous time input and output, but which implements filtering using discrete-time processing:

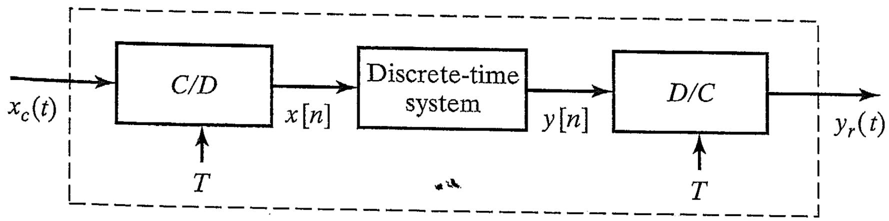
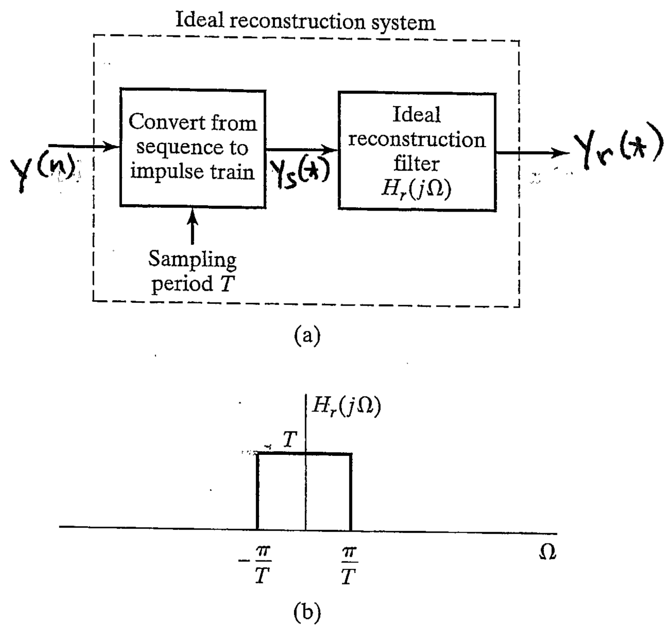


Figure 4.10 Discrete-time processing of continuous-time signals.

An ideal D/C converter using an impulse generator and an ideal analog reconstruction filter is shown in the figure below:



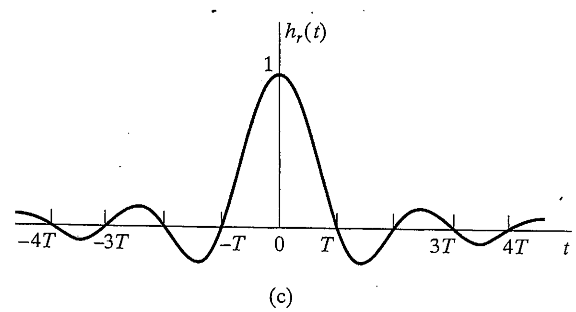


Figure 4.7 (a) Block diagram of an ideal bandlimited signal reconstruction system. (b) Frequency response of an ideal reconstruction filter. (c) Impulse response of an ideal reconstruction filter.

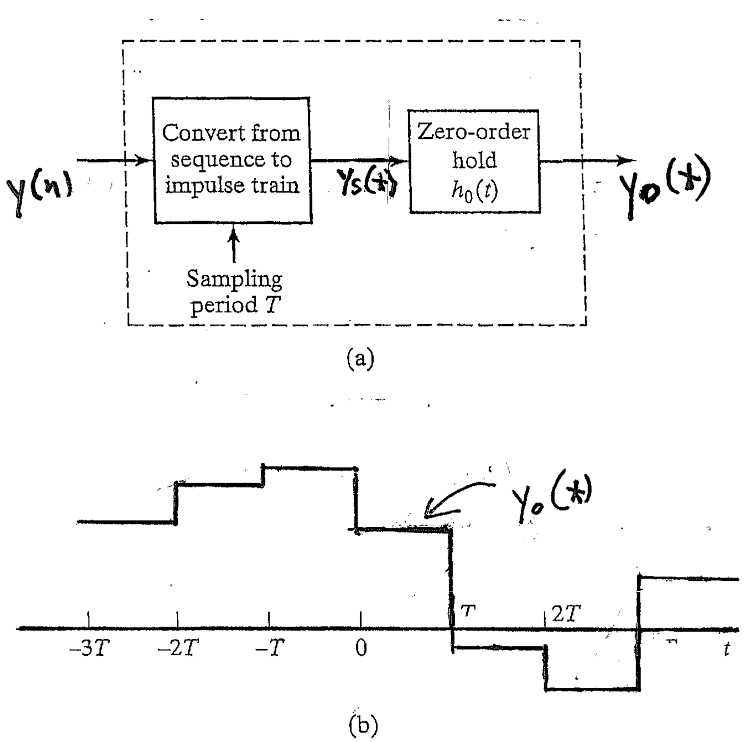
In practice, D/C conversion is typically implemented using a zero-order hold system, followed by an modified analog reconstruction filter.

For this method, the Fourier Transform of the continuous time output is

$$Y(j\Omega)=X(e^{j\Omega T})H(e^{j\Omega T})H_0(j\Omega)\tilde{H}_r(j\Omega)$$

where $H_0(j\Omega)$ is the response of the zero-order hold system and $\tilde{H}_r(j\Omega)$ is the response of the modified analog reconstruction filter.

We have seen that a zero-order hold can be modeled as



where

$$h_0(t)=\begin{cases}1, & 0\le t\le T\\ 0, & \text{all other } t\end{cases}$$

Therefore, $H_0(j\Omega)$ can be found as follows:

$$H_0(j\Omega) = \int_{-\infty}^{\infty} h_0(t)e^{-j\Omega t}\,dt$$

$$= \int_0^T 1 \cdot e^{-j\Omega t}\,dt = \left.\frac{e^{-j\Omega t}}{-j\Omega}\right|_0^T = \frac{1-e^{-j\Omega T}}{j\Omega}$$

$$e^{-j\frac{\Omega T}{2}}\left(\frac{e^{j\frac{\Omega T}{2}} - e^{-j\frac{\Omega T}{2}}}{j\Omega}\right) = e^{-j\frac{\Omega T}{2}}\frac{2\sin\left(\frac{\Omega T}{2}\right)}{\Omega}.$$

The figures below shows the frequency response for the zero-order hold $H_0(j\Omega)$ and the frequency response for the modified reconstruction filter $\tilde{H}_r(j\Omega)$.

The product of these approximates the ideal interpolating filter, $H_r(j\Omega)$.

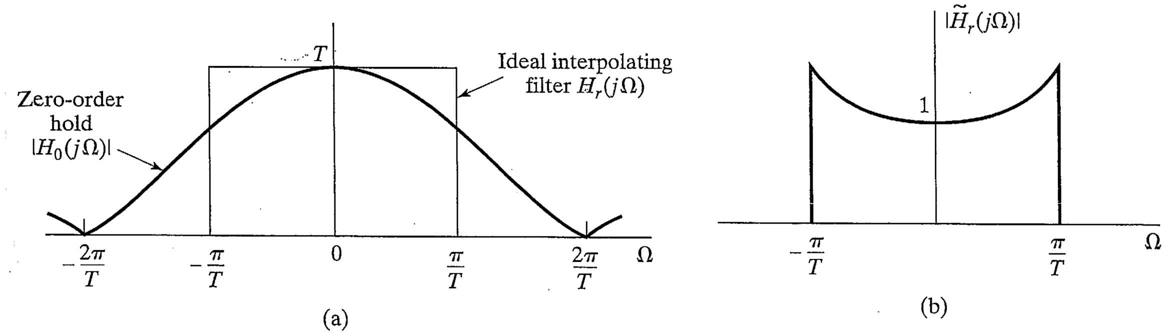


Figure 4.63 (a) Frequency response of zero-order hold compared with ideal interpolating filter. (b) Ideal compensated reconstruction filter for use with a zero-order-hold output.

Note that the above $|\tilde{H}_r(j\Omega)|$ can be expressed mathematically as

$$\tilde{H}_r(j\Omega) = \left| \frac{\left(\frac{\Omega T}{2}\right)}{\sin\left(\frac{\Omega T}{2}\right)} \right|, \quad |\Omega| < \frac{\pi}{T} \qquad \text{and} \qquad \left|\tilde{H}_r(j\Omega)\right| = 0, \quad |\Omega| > \frac{\pi}{T}.$$

<u>Another way to compensate for non-uniform frequency response of the zero-order hold</u> would be to build the compensation into the internal digital filter.

For example, we could modify the original digital filter having frequency response $H(e^{j\Omega T})$ with a modified digital filter having response of

$$\tilde{H}_d(e^{j\Omega T}) = \frac{\Omega T / 2}{\sin(\Omega T / 2)} H(e^{j\Omega T})$$

where $H(e^{j\Omega T})$ represents the desired response of the original digital filter. In this case, the ideal, flat-passband, analog reconstruction filter $H_r(j\Omega)$ could be used for the final step.

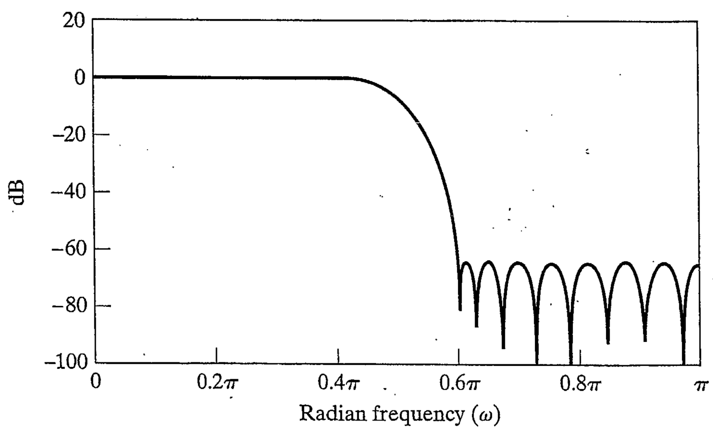
<u>When the desired overall filter is a low-pass filter</u>, we could use the Parks-McClellan algorithm to design a filter having the following target frequency response:

$$\tilde{H}_d(e^{j\omega}) = \begin{cases} \frac{\omega / 2}{\sin(\omega / 2)}, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases}$$

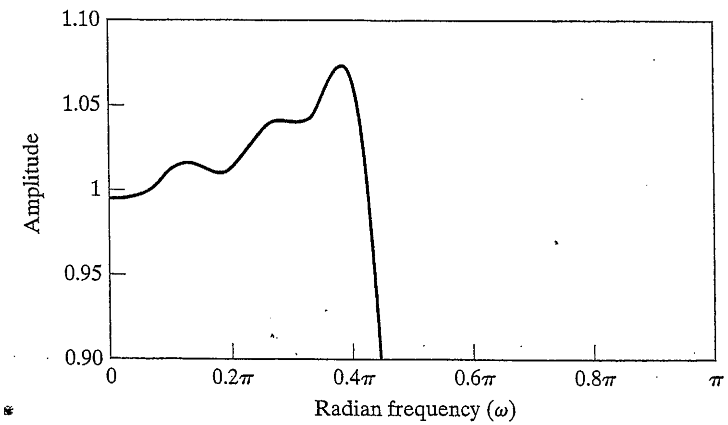
(equation 7.123)

The figure below shows the frequency response for filter of this type, where

$\omega_p = 0.4\pi$ $\omega_s = 0.6\pi$ $\delta_1 = 0.01$ $\delta_2 = 0.001$



(b)



(c)

Figure 7.55 Optimum D/A-compensated lowpass filter for $\omega_p = 0.4\pi, \omega_s = 0.6\pi, K = 10,$ and M=28. (a) Impulse response (b) Log magnitude of the frequency response. (c) Magnitude response in passband.