ECE 8440 - UNIT 19

Properties of the DFT

Linearity

If we form a signal $x_3(n)$ as a linear combination of two other signals, i.e.,

$$x_3(n) = ax_1(n) + bx_2(n) \qquad \text{(equation 8.71)}$$

then

$$X_3[k] = aX_1[k] + bX_2[k].$$

Note: If $x_1(n)$ has length N_1 and $x_2(n)$ has length N_2 , then the length of $x_3(n)$ will be $N_3 = \max[N_1, N_2]$.
(In order for equation 8.71 to be meaningful, all three DFTs, $X_1[k]$, $X_2[k]$, and $X_3[k]$ must be computed using a DFT length N which satisfies $N \geq N_3$.)
Note: Zero-padding of $x_1(n)$ or $x_2(n)$ can be used to satisfy this requirement.

Circular Shift of a Sequence

Consider a finite length sequence $x(n)$ whose DFT is $X[k]$.
We now define a set of DFT coefficients as

$$X_1[k] = e^{-j(2\pi k/N)m} X[k].$$

Now develop a relation between $x_1(n)$ and $x(n)$:

First, form the periodic extension of the above $X_1[k]$ as

$$\tilde{X}_1[k] = X_1[((k))_N] \qquad -\infty \le k \le \infty$$

$$= e^{-j(2\pi((k))_N/N)m} X[((k))_N].$$

Since $e^{-j(2\pi k/N)m}$ is periodic in k with period N, we can write the above expression as

$$= e^{-j(2\pi k/N)m} X[((k))_N].$$

The term $X[((k))_N]$ represents a periodic version of $X(k)$ and can therefore be written as $\tilde{X}(k)$.
Therefore, the previous expression for $\tilde{X}_1(k)$ can be written as

$$\tilde{X}_1(k) = e^{-j(2\pi k/N)m} \tilde{X}(k). \qquad \text{(equation 8.84)}$$

Now, using the "shift property" <u>of the DFS</u>, we can relate $\tilde{x}_1(n)$ and $\tilde{x}(n)$ as follows:

$$\tilde{x}_1(n) = \tilde{x}(n-m) = x[((n-m))_N].$$

Finally, we can obtain $x_1(n)$ as one period of $\tilde{x}_1(n)$:

$$x_1(n) = \begin{cases} \tilde{x}_1(n) = x[((n-m))_N], & 0 \le n \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

Example 8.8 - "Circular Shift" of a Sequence

The effect in the time domain corresponding to multiplying $X(k)$ by the exponential sequence $e^{-j\frac{2\pi km}{N}}$ can be view either as:

1. A linear shift of $\tilde{x}(n)$, the periodic extension of $x(n)$, viewed over the window $0 \le n \le N-1$.
2. A "circular shift" of $x(n)$ over the interval $0 \le n \le N-1$.

This is demonstrated in the figure below.

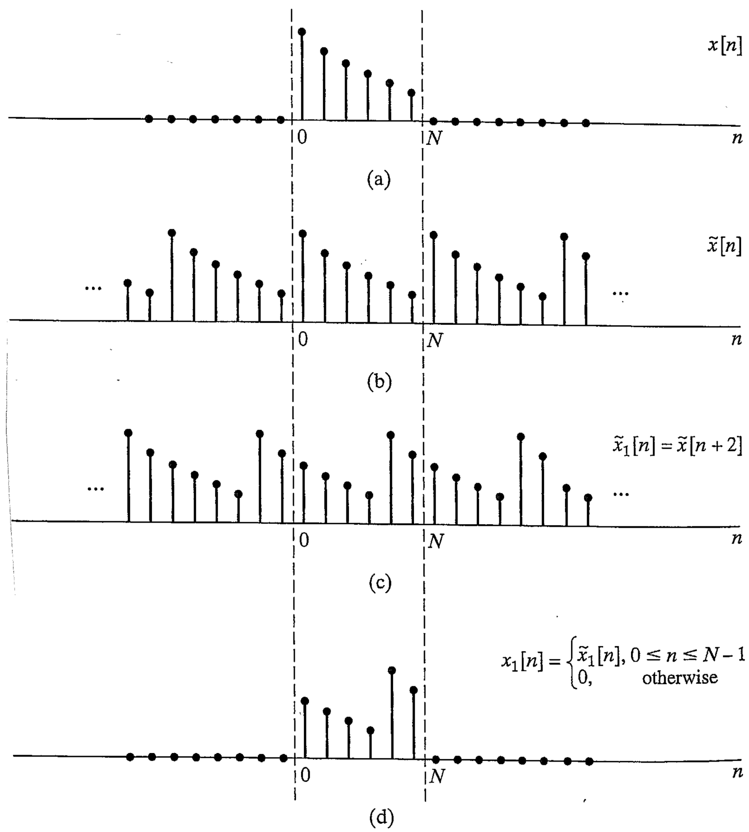


Figure 8.12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear phase factor.

The above circular shift property for the DFT can be expressed compactly as

$$x[((n-m))_N], \quad 0 \le n \le N-1 \overset{DFT}{\leftrightarrow} e^{-j(2\pi k/N)m} X[k] \qquad \text{(equation 8.87)}$$

Duality Property of the DFT

To develop the Duality Property of the DFT, we begin with a non-periodic signal $x(n)$ that extends over $0 \le n \le N-1$. The DFT of this signal is $X(k)$.

Next, we construct $\tilde{x}(n)$, the periodic version of $x(n)$. These signals are related as follows:

$$\tilde{x}(n) = x[((n))_N]. \qquad \text{(equation 8.88a)}$$

Likewise, we construct $\tilde{X}(k)$, the periodic version of $X(k)$ so that

$$\tilde{X}(k) = X[((k))_N]. \qquad \text{(equation 8.88a)}$$

These new sequences are a DFS pair; that is, $\tilde{x}(n) \overset{DFS}{\leftrightarrow} \tilde{X}(k)$

Now, invoking the Duality Property of the DFS, we can write:

$$\tilde{X}(n) \overset{DFS}{\leftrightarrow} N\tilde{x}(-k). \qquad \text{(equation 8.90)}$$

Now define a sequence $\tilde{x}_1(n)$ to be equal to $\tilde{X}(n)$ in the previous equation.

The DFS coefficients of $\tilde{x}_1(n)$ are $\tilde{X}_1(k)$, and, from equation 8.90, can be expressed as

$$\tilde{X}_1(k) = N\tilde{x}(-k).$$

We know that $X_1(k)$, the DFT of $x_1(n)$, is one period of $\tilde{X}_1(k)$. That is,

$$X_1(k) = \begin{cases} N\tilde{x}(-k), & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

This can also be written as

$$X_1(k) = \begin{cases} Nx[((-k))_N], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Note that $X_1(k)$ is the DFT of $x_1(n)$, which was defined as equal to $X(n)$.

Therefore,

$$X(n) \overset{DFT}{\leftrightarrow} Nx[((-k))_N], \qquad 0 \le k \le N-1$$ **(Duality Property of the DFT)**

where

$$x(n) \overset{DFT}{\leftrightarrow} X(k)$$

Example 8.9 – The Duality Relationship for the DFT

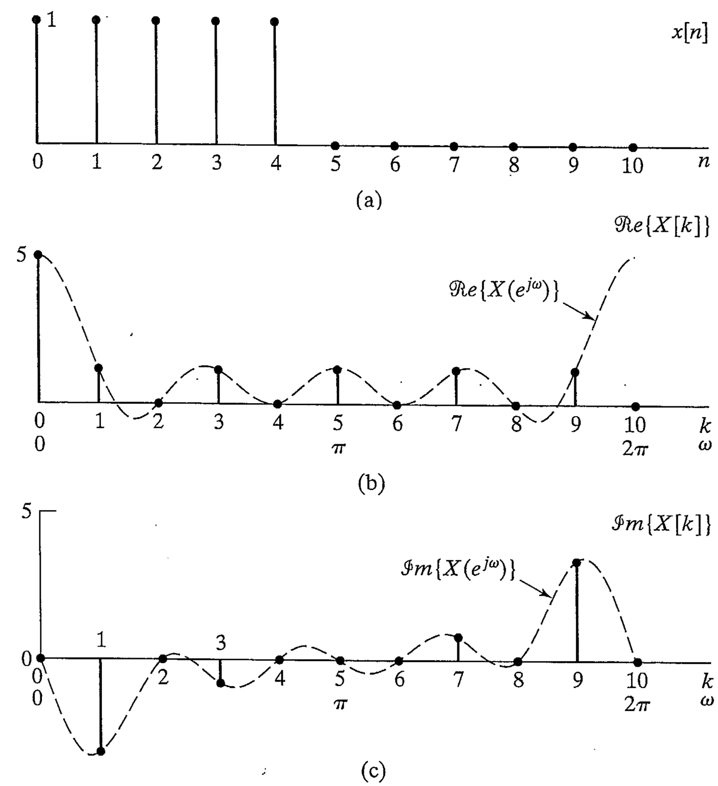
Figure 8.13 demonstrates the Duality Property of the DFT:

-part (a): $x(n)$

-parts (b),(c): real and imaginary parts of $X(k)$

-parts (d),(e): real and imaginary parts of $X(n) = x_1(n)$ (relabeling of axis of parts (b) and(c)) .

-part (f): $DFT[x_1(n)] = 10x[((-k))_{10}]$ (where $N = 10$)



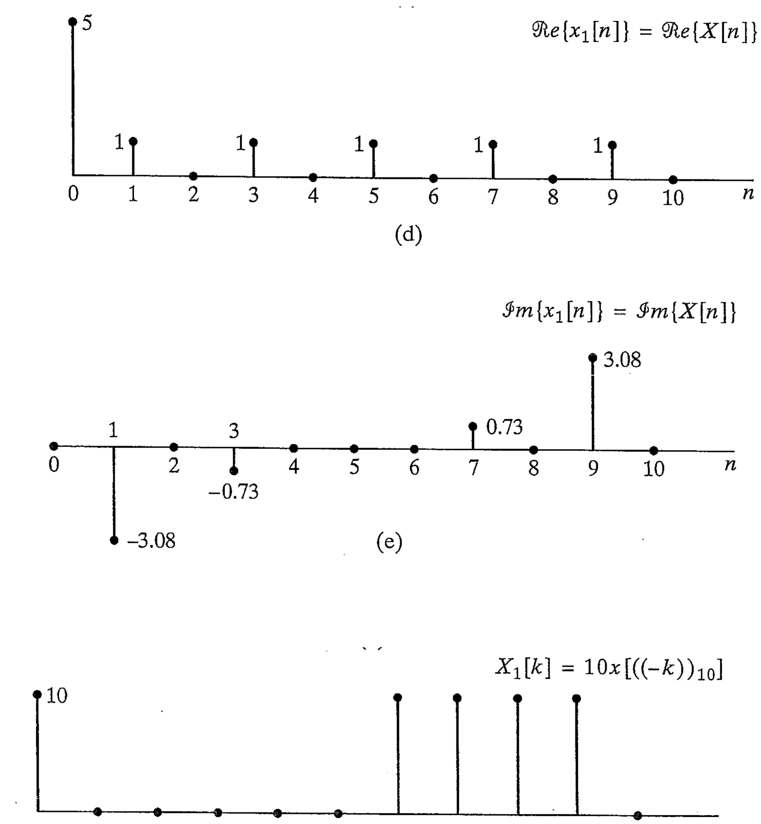


Figure 8.13 Illustration of duality.
(a) Real finite-length sequence $x[n]$.
(b) and (c) Real and imaginary parts of corresponding DFT $X[k]$.
(d) and (e) The real and imaginary parts of the dual sequence $x_1[n] = X[n]$.
(f) The DFT of $x_1[n]$.

Note that part (f) can be formed from part (a) by multiplying by 10, keeping the first term from part (a) fixed, then reversing the order of the remaining terms, to obtain the sequence of part (f).

Section 8.6.5 - Circular Convolution

Consider two finite length signals $x_1(n)$ and $x_2(n)$ with corresponding DFTs $X_1(k)$ and $X_2(k)$. Now define $X_3(k)$ as

$$X_3(k) = X_1(k)X_2(k)$$

The goal is now to express $x_3(n)$ in terms of $x_1(n)$ and $x_2(n)$.

We can do this by considering the periodic extensions $\tilde{x}_1(m)$, $\tilde{x}_2(m)$, $\tilde{x}_3(m)$, $\tilde{X}_1(k)$, $\tilde{X}_2(k)$, and $\tilde{X}_3(k)$ applying DFS properties, then extracting the results over one period, starting at n = 0.

That is, if

$\tilde{X}_3(k) = \tilde{X}_1(k)\,\tilde{X}_2(k)$ then

$$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\tilde{x}_2(n-m)$$

Finally, $x_3(n)$ can be obtained as one period of $\tilde{x}_3(n)$, starting at $n = 0$:

$$x_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\tilde{x}_2(n-m), \quad 0 \le n \le N-1 \qquad \text{(equation 8.112)}$$

This can also be written as

$$x_3(n) = \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \quad 0 \le n \le N-1 \qquad \text{(equation 8.113)}$$

or, since $((m))_N = m$ for $0 \le m \le N-1$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N], \quad 0 \le n \le N-1 \qquad \text{(equation 8.114)}$$

Note that as the above expression is evaluated for increasing values of n, the $x_2[((n-m))_N]$ undergoes a <u>circular shift</u>. This is equivalent to a <u>linear shift</u> of the periodic extension of $x_2(n-m)$ as can be seen in the following figure.

for $n_0 = 1$.

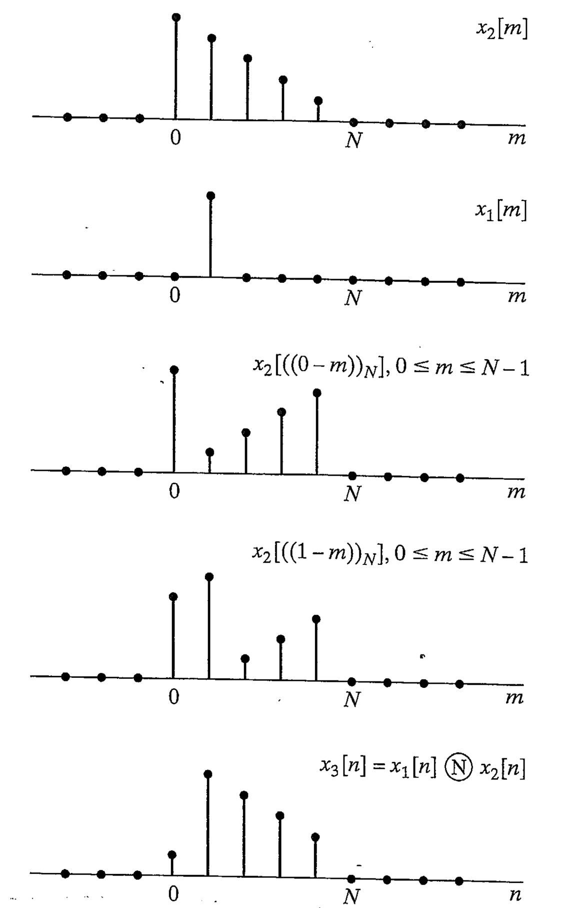


Figure 8.14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse, $x_1[n] = \delta[n-1]$.

Example 8.11 - Circular Convolution of Two Rectangular Pulses

First case:

Consider the following sequences:

$$x_1(n) = x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{other } n \end{cases}$$

Let $X_1(k)$ and $X_2(k)$ be the 6-point DFTs of $x_1(n)$ and $x_2(n)$. Now define $X_3(k)$ as
$X_3(k) = X_1(k)X_2(k)$

Using previous results, we can obtain $x_3(n)$ as the circular convolution of $x_1(n)$ and $x_2(n)$, as shown in the figure below.

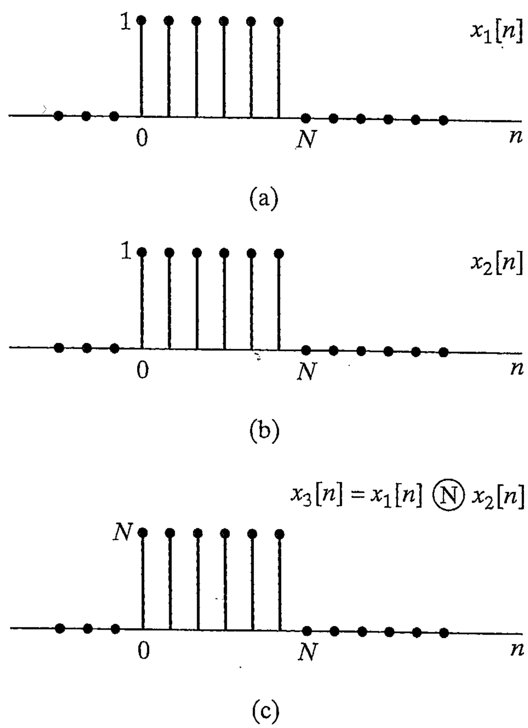


Figure 8.15 N-point circular convolution of two constant sequences of length N.

Second case:

If we use the same $x_1(n)$ and $x_2(n)$ used above, but now use a DFT of length N=12 in obtaining $X_1(k)$ and $X_2(k)$, the sequence $x_3(n)$ that is obtained as the inverse DFT of $X_3(k) = X_1(k)X_2(k)$ is shown in the following figure:

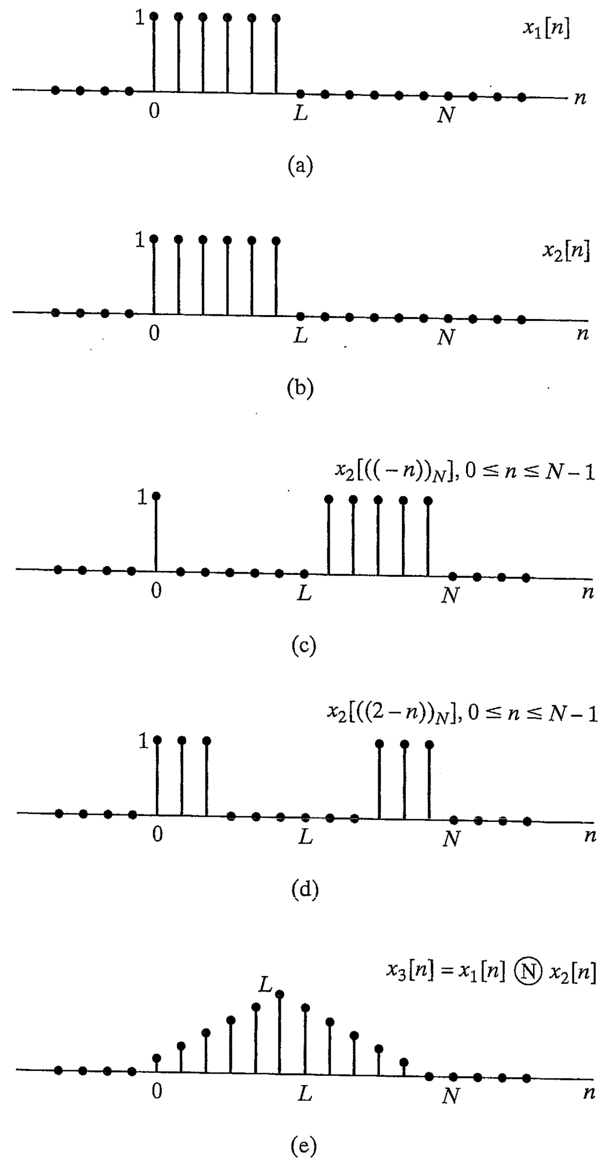


Figure 8.16 2*L*-point circular convolution of two constant sequences of length *L*.

Note that by "zero-padding" $x_1(n)$ and $x_2(n)$ before applying the DFT, the inverse DFT of the product $X_3(k) = X_1(k)X_2(k)$, the resulting $x_3(n)$ is equivalent to the linear convolution of $x_1(n)$ and $x_2(n)$.

Section 8.7.3 - Implementing Linear Time-Invariant Systems Using the DFT

Consider an L-point input signal $x(n)$ applied to an FIR system whose unit sample response $h(n)$ has length P. Then the output $y(n)$, which is the linear convolution of $x(n)$ and $h(n)$ has maximum duration of L+P-1.

In order to obtain L+P-1 outputs using the DFT approach, each DFT must therefore have length L +P-1. We can achieve this by padding each input segment with P-1 zeros and padding h(n) with L-1 zeros before applying the DFT.

Typically, the signal to be filtered is too long to process with a single DFT operation. In this case, we segment the signal into subsections of length L and perform what is called block convolution. There are several ways to do this, as shown below:

Overlap Add Method

First, segment the input into sections of length L , where L is chosen so that the DFT length of L+P-1 is manageable. Representation of x(n) in terms of its segments is given by

$$x(n) = \sum_{r=0}^{\infty} x_r(n - rL) \qquad \text{(equation 8.140)}$$

where

$$x_r(n) = \begin{cases} x(n+rL), & 0 \le n \le L\text{-}1 \\ 0, & \text{otherwise} \end{cases} \qquad \text{(equation 8.141)}$$

For example, consider the signal x(n) in the figure below:

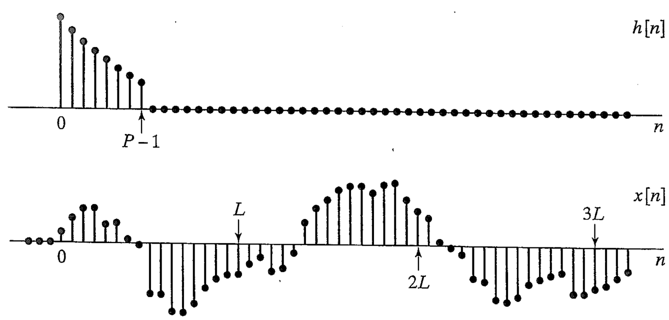
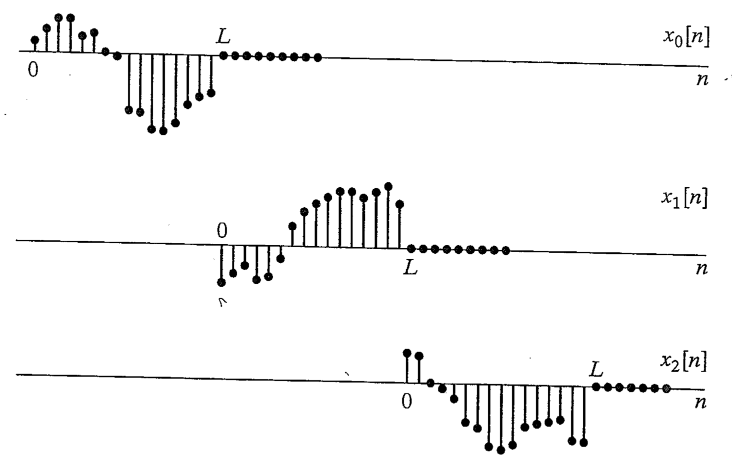


Figure 8.22 Finite-length impulse response $h[n]$ and indefinite-length signal $x[n]$ to be filtered.

Segmentation of the above signal into L-length blocks is shown in the following figure:



(a)

If the first input block filtered using the DFT method, a total of L+P-1 output points will be generated. Note that the last P-1 of these point will correspond to time indices larger that the highest time index of the first input block. These last P-1 output points of the first block will have to be added to the first P-1 outputs from processing the second input block, in order to completely represent the output for $L \le n \le L+P-2$

The same adjustment will have to be made for each successive input block that is processed. (Add the first P-1 outputs for each newly processed segment to the last P-1 outputs for the previous segment.)

This procedure is depicted in the figure below:

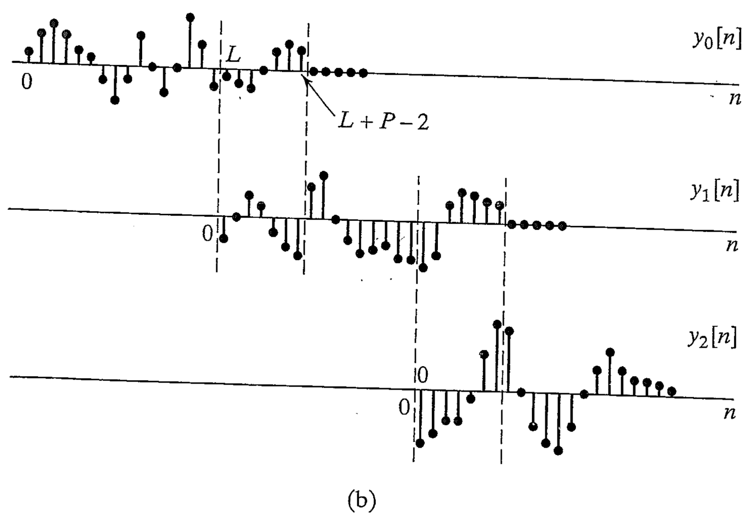


Figure 8.23 (a) Decomposition of $x[n]$ in Figure 8.22 into nonoverlapping sections of length L. (b) Result of convolving each section with $h[n]$.

Overlap Save Method

Rather than adding the overlapped outputs corresponding to input segments which are non-overlapped and zero padded, the Overlap Save method overlaps the input segments (without zero-padding), then discards the first part of each output block. (No adding of outputs is performed.) Details of the Overlap Save approach are shown below:

First, define the input segments as

$$x_r(n) = x[n + r(L - P + 1) - (P - 1)], \quad 0 \le n \le L\text{-}1$$

$$= x[n + r(L - P + 1) - P + 1], \quad 0 \le n \le L\text{-}1 \qquad \text{(equation 8.144)}$$

(Note that segments will overlap by P-1 points.)

Since input blocks are not zero-padded for this method, the first P-1 output points will not correspond to linear convolution. (Two periods of h(n) will overlap with one period of the input segment in the generation of the first P-1 outputs). Therefore, the first P-1 outputs of each output segment are discarded.

By overlapping the input segments by P-1 points, we ensure that there are no gaps in generating the correct output corresponding to linear convolution.

If $y_{rp}(n)$ denotes the original output of each DFT-implemented circular convolution, the additional processing of the output segments necessary to obtain $y(n)$ is shown below:

$$y_r(n) = \begin{cases} y_{rp}(n), & P-1 \le n \le L-1 \\ 0, & 0 \le n \le P-2 \end{cases}$$

and

$$y(n) = \sum_{r=0}^{\infty} y_r[n - r(L-P+1) + (P-1)] \qquad \text{(equation 8.145)}$$

The following figure demonstrates implementation of the overlap save method.

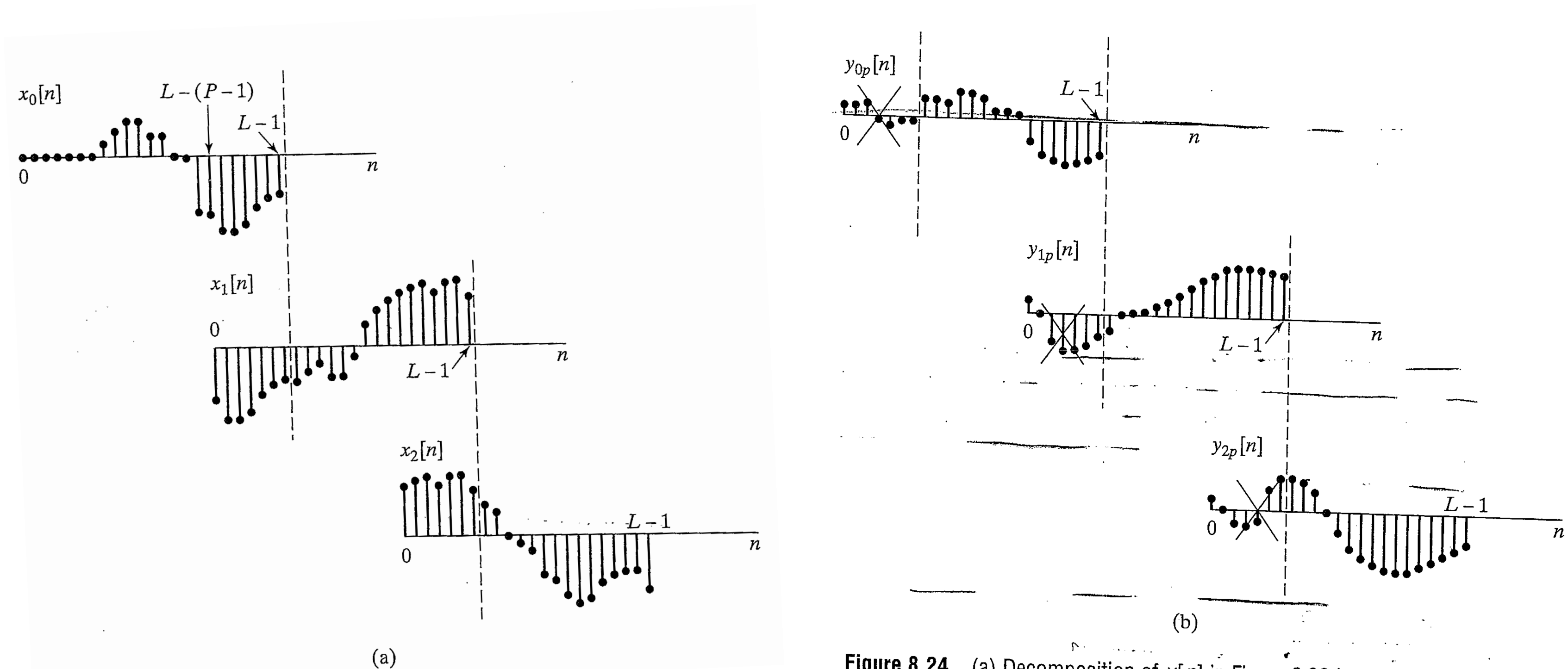


Figure 8.24 (a) Decomposition of $x[n]$ in Figure 8.22 into overlapping sections of length L. (b) Result of convolving each section with $h[n]$. The portions of each filtered section to be discarded in forming the linear convolution are indicated.