<u>The Discrete Cosine Transform</u>

- similar to the DFT
- has advantages over the DFT in applications involving data compression (e.g., coding of speech and image signals)
- can be computed by using an algorithm that applies the DFT to a modified input.

Before defining the Discrete Cosine Transform, consider the general representation of a transform of a finite length time domain signal $x(n)$:

$$A(k) = \sum_{n=0}^{N-1} x(n)\phi_k^*(n) \qquad \text{(analysis equation)} \qquad \text{(equation 8.147)}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} A(k)\phi_k(n) \qquad \text{(synthesis equation)} \qquad \text{(equation 8.148)}$$

The sequences $\phi_k(n)$ in the above expressions are called <u>basis sequences</u> and are orthogonal to each other, so that they satisfy:

$$\frac{1}{N}\sum_{k=0}^{N-1} \phi_k(n)\,\phi_m^*(n) = \begin{cases} 1, & m = k \\ 0, & m \neq k. \end{cases}$$

<u>For the DFT</u>, the $\phi_k(n)$ are the periodic complex exponential sequences:

$\phi_k(n) = e^{j2\pi kn/N}$.

There are some orthogonal transforms which yield a <u>real-valued</u> $A(k)$ sequence when $x(n)$ is real valued. These include:

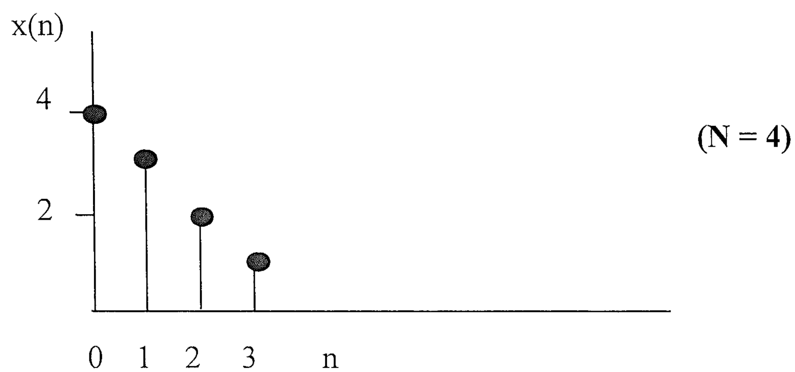
- Haar Transforms
- Hadamard Transforms
- Hartley Transforms
- **Discrete Cosine Transform**

<u>The Discrete Cosine Transform</u> uses cosine functions for the $\phi_k(n)$ basis functions.

<u>Important properties of cosine functions</u> when used as basis functions for a transform:

- periodic (like the exponentials used by the DFT)
- even symmetry

To begin the development of the DCT, consider the 4-point signal shown below:



Each of the 8 versions of the DCT is based on extending an N-point sequence into a periodic sequence which has certain symmetry properties, as described below:

Type-1 Periodic Extension

The first version of the DCT, called DCT-1, is based on extending **x(n)** into the periodic signal $\tilde{x}_1(n)$ using "Type-1 periodic extension." For the **x(n)** above , the Type-1 periodic extension is shown below:

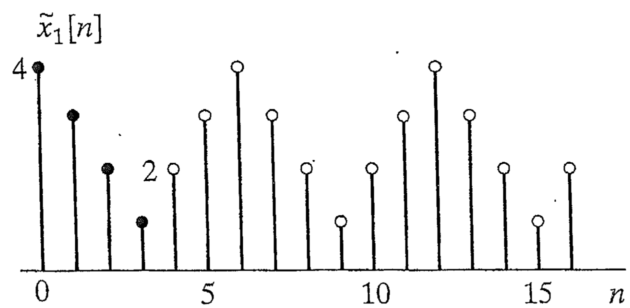


Figure 8.25 (a)

Center of even symmetry:

- The last point of x(n) (n = N-1, where N is the number of points in x(n))

Period of $\tilde{x}_1(n)$: $2N - 2 = 6$

Type-2 Periodic Extension

The second version of the DCT is based on extending $x(n)$ into the periodic signal $\tilde{x}_2(n)$ using "Type-2 periodic extension." For the above $x(n)$, the Type-2 periodic extension is shown below:

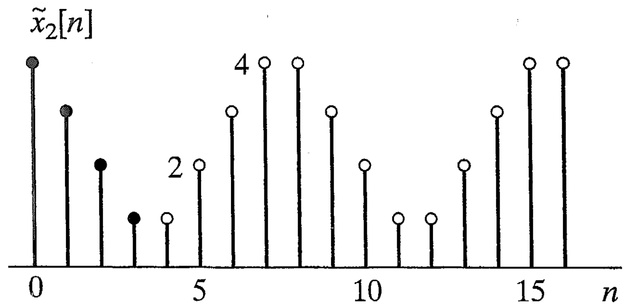


Figure 8.25 (b)

Center of even symmetry:

- Half-sample point after last sample ($"n" = N - \frac{1}{2}$, where N = number of points in x(n)).

Period of $\tilde{x}_2(n)$: $2N = 8$

Type-3 Periodic Extension

The third version of the DCT is based on extending x(n) into the periodic signal $\tilde{x}_3(n)$ using "Type-3 periodic extension." For the above x(n), the Type-3 periodic extension is shown below:

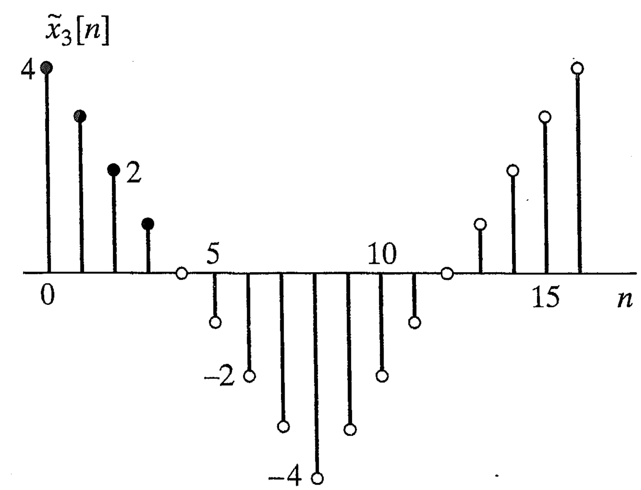


Figure 8.25 (c)

Center of odd symmetry

- Point after the last point of x(n) (n = N, where N = length of x(n))
  (Note that x(N) must be 0.)

Center of even symmetry:

- The point n = 2N

Period of $\tilde{x}_3(n)$: 4N = 16

Type-4 Periodic Extension

Still another version of the DCT, called DCT-4, is based on extending x(n) into the periodic signal $\tilde{x}_4(n)$ using "Type-4 periodic extension." For the above x(n), the Type-4 periodic extension is shown below:

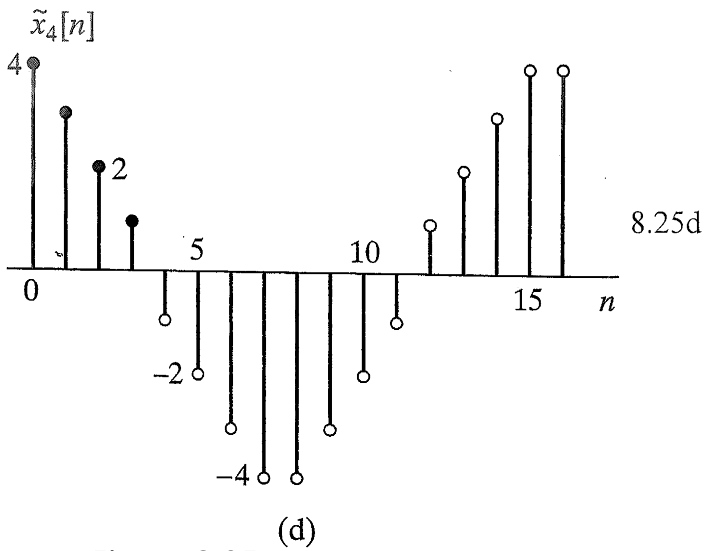


Figure 8.25

Center of odd symmetry:

- Half-sample point after last point of x(n) ("n" = $N - \frac{1}{2}$, where N = number of points in x(n))

Center of even symmetry:

- The point "n" = $2N + \frac{1}{2}$, where N = number of points in x(n))

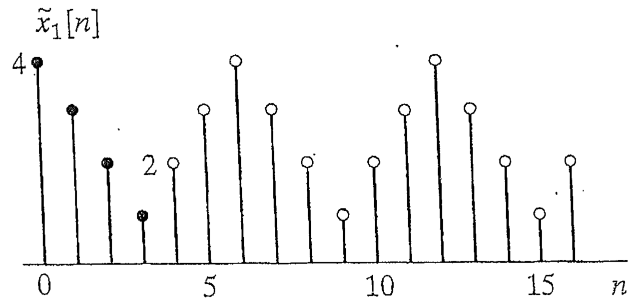
Period of $\tilde{x}_4(n)$: $4N = 16$

<u>Four versions of the DCT are closely related to the extended signals shown above</u>.
These are:

DCT - 1 $\leftrightarrow$ $\tilde{x}_1(n)$
DCT - 2 $\leftrightarrow$ $\tilde{x}_2(n)$
DCT - 3 $\leftrightarrow$ $\tilde{x}_3(n)$
DCT - 4 $\leftrightarrow$ $\tilde{x}_4(n)$

<u>Definition of DCT-1</u>

First, note from the plot of $\tilde{x}_1(n)$, shown again below, that we could create a signal <u>similar to</u> $\tilde{x}_1(n)$ by adding a periodic extension of x(n) (after appending N-2 0's) to a periodic extension of x(-n) (also with N-2 appended 0's). Note that both extensions have the same period of 2N-2.



(a)

(Repeat of figure shown on a previous slide.)

(The resulting signal would be same as $\tilde{x}_1(n)$ except that the first and last points in x(n) would overlap with the first and last points in x(-n) when both are periodically extended, as described above.)

To compensate for the "double value" terms we define a pre-scaled signal $x_\alpha(n)$ as

$$x_\alpha(n) = \alpha(n)x(n)$$

where

$$\alpha(n) = \begin{cases} \frac{1}{2}, & n = 0 \text{ and } N-1 \\ 1, & 1 \le n \le N-2 \end{cases}$$

Then, adding the periodic extensions of $x_\alpha(n)$ and $x_\alpha(-n)$ with period = $2N-2=6$ will create the signal we defined earlier as $\tilde{x}_1(n)$. This can be formally expressed as

$$\tilde{x}_1(n) = x_\alpha[((n))_{2N-2}] + x_\alpha[((-n))_{2N-2}] \qquad \text{(equation 8.150)}$$

We now formally define the DCT-1 of an N-point signal and the inverse DCT-1 by the following two equations:

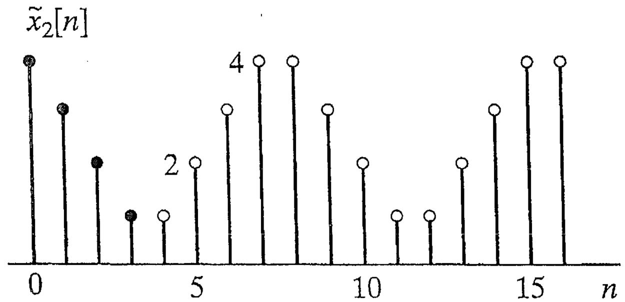
$$X^{c1}(k) = 2\sum_{n=0}^{N-1} \alpha(n)x(n)\cos\left(\frac{\pi kn}{N-1}\right), \qquad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N-1}\sum_{k=0}^{N-1} \alpha(k)X^{c1}(k)\cos\left(\frac{\pi kn}{N-1}\right), \qquad 0 \le k \le N-1$$

where $\alpha(k)$ is the same sequence as used in defining $x_\alpha(n)$.

Definition of DCT-2

The DCT-2 transform is related to $\tilde{x}_2(n)$, the second version of a periodic sequence based on x(n). Note from the plot of $\tilde{x}_2(n)$ that we could create $\tilde{x}_2(n)$ by adding a periodic extension of $x(n)$ (with period = 2N) to a periodic extension of x(-n-1), also with period = 2N.



(b)

(Repeat of figure shown on a previous slide.)

The resulting signal would be exactly the same as $\tilde{x}_2(n)$ since there are no overlaps involved when the two contributions are added. We can formally express $\tilde{x}_2(n)$ in terms of $x(n)$ as follows:

$$\tilde{x}_2(n) = x[((n))_{2N}] + x[((-n-1))_{2N}] \qquad \text{(equation 8.154)}$$

We now formally define the DCT-2 and its inverse by the following two equations:

$$X^{c2}(k) = 2\sum_{n=0}^{N-1} x(n)\cos\left(\frac{\pi k(2n+1)}{2N}\right), \qquad 0 \le k \le N\text{-}1$$

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} \beta(k) X^{c2}(k)\cos\left(\frac{\pi k(2n+1)}{2N}\right), \qquad 0 \le n \le N\text{-}1$$

The weighting function $\beta(k)$ used in the above expression for the inverse DCT-2 is defined as follows:

$$\beta(k) = \begin{cases} \frac{1}{2}, & k = 0 \\ 1, & 1 \le k \le N-1. \end{cases}$$

The sequences $X^{c1}(k)$ and the $X^{c2}(k)$ obtained for the 4-point signal $x(n)$ used to form the $\tilde{x}_1(n)$ and $\tilde{x}_2(n)$ described above are shown below:

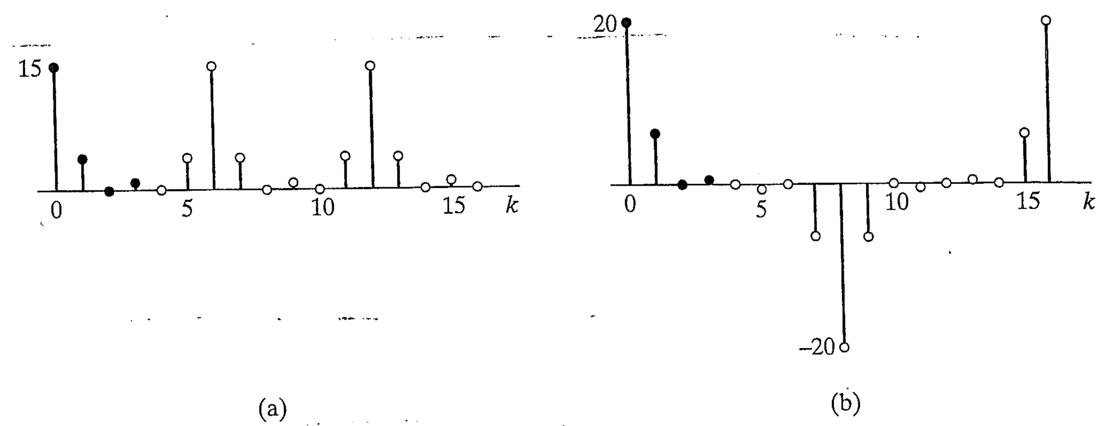


Figure 8.26 DCT-1 and DCT-2 for the four-point sequence used in Figure 8.25. (a) DCT-1. (b) DCT-2.

Although $X^{c1}(k)$ and $X^{c2}(k)$ are normally evaluated only for the range $0 \le k \le N-1$ (for the above example, N = 4), if we do evaluate them outside this range, they exhibit symmetry properties.

However, it is not always the same type of symmetry associated the periodically extended time signal it is related to. That is, $X^{c1}(k)$ has Type 1 symmetry, like $\tilde{x}_1(n)$. However, $X^{c2}(k)$ has Type 3 symmetry, not the Type 2 symmetry of $\tilde{x}_2(n)$.

<u>Relation Between the DCT-1 and the DFT</u>

Consider <u>one period</u> of the periodic time signal associated with DCT-1:

$x_1(n) = x_\alpha[((n))_{2N-2}] + x_\alpha[((-n))_{2N-2}] = \tilde{x}_1(n) \quad n = 0,1,\ldots,2N\text{-}3$ (eqn. 8.161)

where as before $x_\alpha(n)$ is the original N-point sequence $x(n)$ with its endpoints multiplied by $\frac{1}{2}$.
Note that the number of points in $x_1(n)$ is $2N-2$.

Now take the $2N-2$ point <u>DFT</u> of $x_1(n)$:

$X_1(k) = X_\alpha(k) + X_\alpha^*(k) = 2\mathrm{Re}\{X_\alpha(k)\}, \qquad k=0,1,\ldots,2N\text{-}3$

(See property 10 of the DFT (in Table 8.1) and property 10 of the DFS (in Tables 8.2)
where $X_\alpha(k)$ is the 2N -2 point DFT of $x_\alpha(n)$ after $x_\alpha(n)$ is zero-padded to extend its length from N to 2N-2 .

Finally, expressing $\mathrm{Re}\{X_\alpha(k)\}$ using cosines, we can write the above expression for $X_1(k)$ as

$$X_1(k) = 2\mathrm{Re}\{X_\alpha(k)\} = 2\sum_{n=0}^{N-1}\alpha(n)x(n)\cos\left(\frac{2\pi kn}{2N-2}\right) = 2\sum_{n=0}^{N-1}\alpha(n)x(n)\cos\left(\frac{\pi kn}{N-1}\right), \qquad 0 \le k \le 2N-1.$$

Note that the right side of the above expression is the same expression we used to define $X^{c1}(k)$, the DCT-1 of x(n).

Therefore, we can find the DCT-1 of an N-point signal as follows:

1. Form $x_\alpha(n) = \alpha(n)x(n) \qquad 0 \le n \le N\text{-}1$
2. Form $x_1(n) = x_\alpha[((n))_{2N-2}] + x_\alpha[((-n))_{2N-2}] = \tilde{x}_1(n) \quad n = 0,1,\ \dots\ ,2N\text{-}3$
3. Take the 2N-2 point DFT of $x_1(n)$ to get $X_1(k)$
4. Extract the first N terms of $X_1(k)$ to form $X^{c1}(k)$. (This provides terms for $0 \le k \le N\text{-}1$.)

OR (the simpler approach)

1. Form $x_\alpha(n) = \alpha(n)x(n) \qquad 0 \le n \le N\text{-}1$
2. "Zero-pad" $x_\alpha(n)$ with N-2 zeros to create a sequence over the range $0 \le n \le 2N\text{-}3$.
3. Take the 2N-2 point DFT of the above sequence to get $X_\alpha(k)$.
4. Form $X^{c1}(k)$ as twice the real part of $X_\alpha(k)$ for $0 \le k \le N\text{-}1$.

Inverse DCT-1

To find the Inverse DCT-1, we start with the knowledge that the first N terms of $X_1(k)$ are equal to to $X^{c1}(k)$.

Then we use the symmetry property for the DFT of real signals: $X_1(k) = X_1^*(2N-2-k)$

and the fact that $X^{c1}(k)$ is real valued to form $X_1(k)$ from $X^{c1}(k)$ as follows:

$$X_1(k) = \begin{cases} X^{c1}(k), & k = 0,\dots,N\text{-}1 \\ X^{c1}(2N-2-k) & k = N,\dots,2N\text{-}3 \end{cases}$$

Then, applying the formula for evaluating 2N-2 point inverse DFT, we can express $x_1(n)$ as

$$x_1(n) = \frac{1}{2N-2}\sum_{k=0}^{2N-3} X_1(k)e^{j2\pi kn/(2N-2)}, \qquad n = 0, \ldots, 2N-3.$$

Finally, we can obtain $x(n)$ by extracting the first N points of $x_1(n)$. That is,

$$x(n) = x_1(n), \qquad n=0, \ldots, N-1$$

Relation of the DCT-2 and the DFT

Consider one period of the periodic signal associated with the DCT-2:

$$x_2(n) = x[((n))_{2N}] + x[((-n-1))_{2N}] = \tilde{x}_2(n) \quad n=0,1,\ldots,2N-1 \qquad \text{(equation 8.166)}$$

where $x(n)$ is the original N-point signal. Now take the 2N point DFT of $x_2(n)$:

$$X_2(k) = X(k) + X^*(k)e^{j2\pi k/(2N)}, \qquad k=0,1,\ldots,2N-1$$

$$= e^{j\pi k/(2N)}\left(X(k)e^{-j\pi k/(2N)} + X^*(k)e^{j\pi k/(2N)}\right)$$

$$= e^{j\pi k/(2N)} 2\,\mathrm{Re}\left\{X(k)e^{-j\pi k/(2N)}\right\}$$

where $X(k)$ is the 2N-point DFT of the sequence formed by zero-padding $x(n)$ to extend its length from N to 2N.

Note that $\mathrm{Re}\left\{X(k)e^{-j\pi k/(2N)}\right\}$ can be expressed as:

$$\mathrm{Re}\left\{X(k)e^{-j\pi k/(2N)}\right\} = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{\pi k(2n+1)}{2N}\right).$$

Therefore, we can write the above expression for $X_2(k)$ as

$$X_2(k) = e^{j\pi k/(2N)} 2\sum_{n=0}^{N-1} x(n)\cos\left(\frac{\pi k(2n+1)}{2N}\right) \qquad k=0,1,\ldots,2N-1$$

Comparing this with the expression we used to define $X^{c2}(k)$, (see slide 9) we can now write

$$X^{c2}(k) = e^{-j\pi k/(2N)} X_2(k), \qquad k=0,1,\ldots,N-1$$

Therefore, we can find the DCT-2 of $x(n)$ by the following steps:

1. Take the 2N-point DFT of $x_2(n) = x[((n))_{2N}] + x[((-n-1))_{2N}]$, where $x_2(n)$ is one period of $\tilde{x}_2(n)$.
2. Multiply the first N terms of the output of step 1 by $e^{-j\pi k/(2N)}$ to obtain $X^{c2}(k)$ for , $k=0,1,\ldots,N-1$

Inverse DCT-2

To find the Inverse DCT-2, we start with the previous result that $X^{c2}(k)$ is equal to the first N terms of $X_2(k)e^{-j\pi k/(2N)}$. We also use the fact that $X^{c2}(k)$ has Type 3 symmetry. i.e.,

$$X^{c2}(2N-k) = -X^{c2}(k), \qquad k = 0, 1, \ldots, 2N-1$$

(This symmetry property can be shown using the definition fo the DCT-2 in eqn. 8.155).

Combining the above two facts, we can write:

$$X_2(k) = \begin{cases} X^{c2}(0), & k = 0 \\ e^{j\pi k/(2N)} X^{c2}(k) & k = 1, \ldots, N-1 \\ 0 & k = N \\ -e^{j\pi k/(2N)} X^{c2}(2N-k) & k = N+1, N+2, \ldots, 2N-1 \end{cases} \qquad \text{equation 8.174}$$

Then, using the formula for evaluating the 2N-point inverse DFT, we can express $x_2(n)$ as

$$x_2(n) = \frac{1}{2N} \sum_{k=0}^{2N-1} X_2(k) e^{j2\pi kn/(2N)}, \qquad n=0,\ldots,2N-1 \; . \qquad \text{equation 8.175}$$

Finally, we can obtain $x(n)$ by extracting the first N points of $x_2(n)$. That is,

$$x(n) = x_2(n), \qquad n = 0, \ldots, N-1.$$

<u>Energy Compaction Property of the DCT-2</u>

More of the energy of a signal is represented by the low indices of its DCT-2 than by the low indices of its DFT, as shown in the example below:

<u>Example 8.13</u>

Consider a signal of the form:

$$x(n) = a^n \cos(\omega_0 n + \phi), \qquad n = 0, \ldots, N-1.$$

This signal is plotted below for the following set of parameter values:

$a = .9,\ \omega_0 = .1\pi,\ \phi = 0, \text{and } N = 32.$

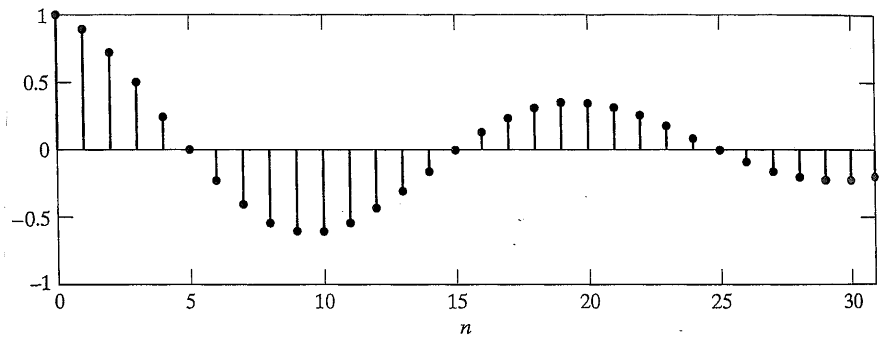


Figure 8.27 Test signal for comparing DFT and DCT.

The first 16 terms of the 32-point DFT are shown in parts (a) and (b) of the figure below. (The second 16 terms in each case are conjugate symmetric pairs with the first 16 terms, so they contain no additional information) Part (c) of the figure shows the 32-point DCT-2.

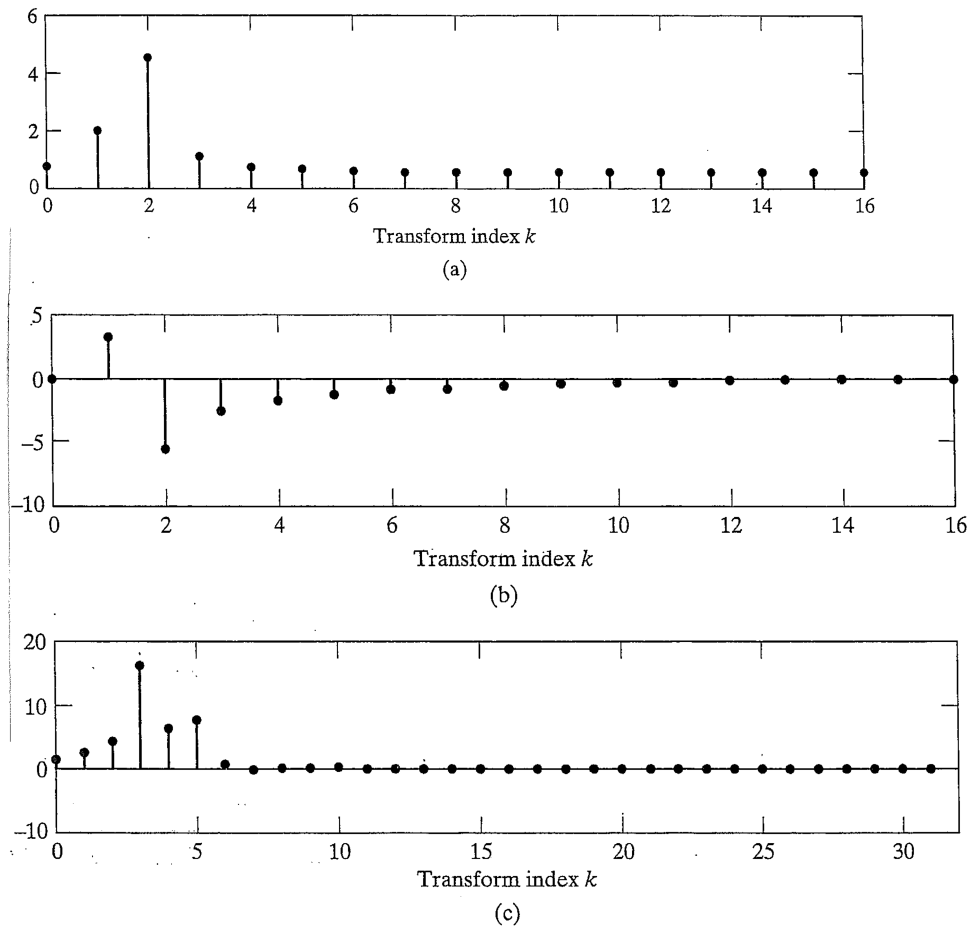


Figure 8.28 (a) Real part of N-point DFT; (b) Imaginary part of N-point DFT; (c) N-point DCT-2 of the test signal plotted in Figure 8.27.

Note that more of the "energy" of the DCT-2 is contained in the lower index terms, than for the case of the DFT.

To quantify the comparison of the energy compaction property of the DCT-2 with that of the DFT, first define a term that represents the approximation of the time-domain signal x(n) using the DFT synthesis equation after removing m of the higher frequency terms from X(k). That is,

$$x_m^{dft}(n) = \frac{1}{N}\sum_{k=0}^{N-1} T_m(k)X(k)e^{j2\pi kn/N}, \qquad n=0, \ldots, N-1$$

where

$$T_m(k) = \begin{cases} 1, & 0 \le k \le (N-1-m)/2 \\ 0. & (N+1-m)/2 \le k \le (N-1+m)/2 \\ 1, & (N+1+m)/2 \le k \le N-1 \end{cases}$$

(sets m "middle-indexed" terms to 0)

Note that when m = 1, x(N/2) is removed.
For m = 3, X(N/2-1), x(N/2), and X(N/2+1) are removed.
For m = 5, x(N/2-2), x(N/2-1), x(N/2), x(N/2+1), and x(N/2+2) are removed.

For comparison, the synthesis of x(n) using a truncated set of DCT-2 coefficients in which m coefficients have been removed can be represented as

$$x_m^{dct}(n) = \frac{1}{N}\sum_{k=0}^{N-1-m} \beta(k)X^{c2}(k)\cos\left(\frac{\pi k(2n+1)}{2N}\right), \quad n=0, \ldots, N-1$$

To compare the errors in synthesizing x(n) using truncated sets of DFT and DCT-2 coefficients, we define the following error functions:

$$E^{dft}(m)=\frac{1}{N}\sum_{n=0}^{N-1}|x(n)-x_m^{dft}(n)|^2$$ and

$$E^{dct}(m)=\frac{1}{N}\sum_{n=0}^{N-1}|x(n)-x_m^{dct}(n)|^2$$

The following figure plots both of the above error functions for the previously described signal, which is

$$x(n)=.9^n\cos(.1\pi n),\qquad n=0,\ldots,31$$

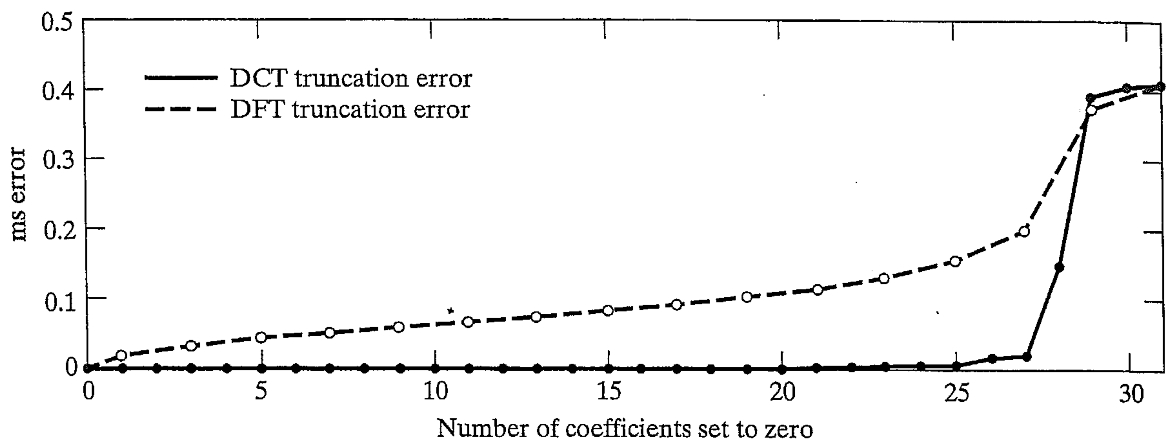


Figure 8.29 Comparison of truncation errors for DFT and DCT-2.

Because of the good energy compaction property of the DCT-2 as compared to the DFT, the DCT-2 is very effective for use in data compression applications (e.g., speech and image processing.)