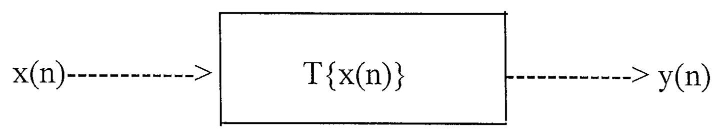
ECE 8440 Unit 21

Cepstral Analysis and Homomorphic Deconvolution

Recall the general representation of a linear system:



A linear system exhibits the following superposition property:

$$T\{x_1(n) + x_2(n)\} = T\{x_1(n)\} + T\{x_2(n)\}$$

and also

$$T\{cx(n)\} = cT\{x(n)\}$$

where c is any scalar.

We now generalize to the case of homomorphic systems, which obey a general principal of superposition:

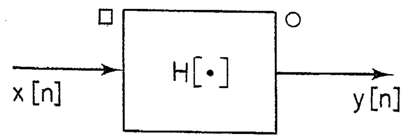
$$H\{x_1(n) \,\square\, x_2(n)\} = H\{x_1(n)\} \circ H\{x_2(n)\}$$

where " $\square$ " is an operation for combining inputs and " $\circ$ " is an operation for combining outputs. The homomorphic system also satisfies

$$H\{c : x(n)\} = c \,\iota\, H\{x(n)\}$$

where " :" is an operation for combining inputs with scalers and " ι " is an operation for combining outputs with scalars.

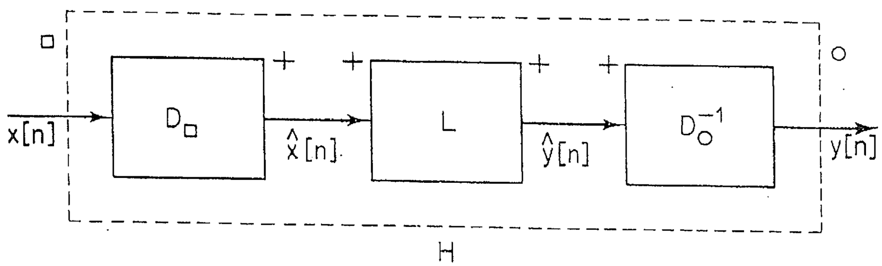
The following figure provides a block diagram representation for a homomorphic system:



Representation of a system satisfying generalized superposition with input operation □, output operation o, and system transformation *H*.

For the special case where □ and o represent addition and : and ι represent multiplication, the system is linear.

The following is called a "canonic" representation of a homomorphic system:



Canonic representation of homomorphic systems.

The subsystem $D_\square$ is called the <u>characteristic system for the operation □</u>, since it converts inputs combined by the operation □ to a signal whose components are combined by addition.

The subsystem $D_\circ^{-1}$ is called the <u>inverse characteristic system for the operation o</u> since it converts inputs combined by addition to a signal whose components are combined by the operation o .

Consider the canonic representation for a system for which the input operation is convolution (*) and the output operation is also convolution (*).

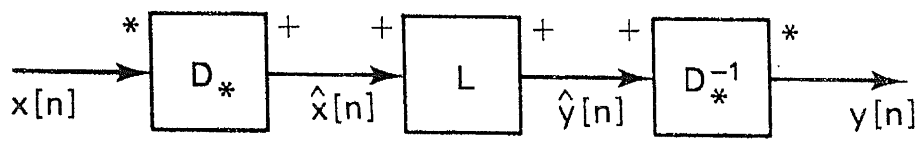


Figure 13.6 Canonic form for homomorphic systems where inputs and corresponding outputs are combined by convolution.

If $\quad x(n) = x_1(n) * x_2(n)$

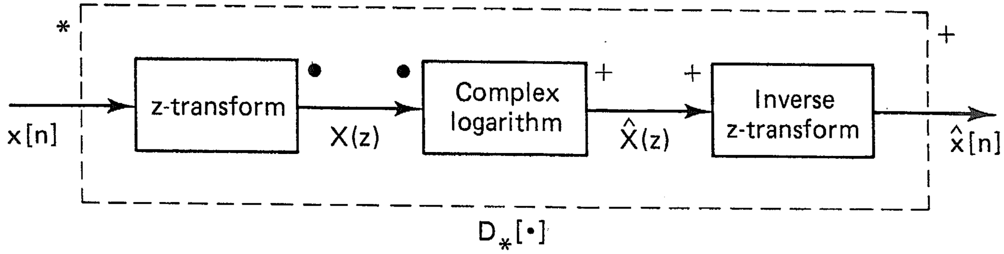
then $\hat{x}(n) = \hat{x}_1(n) + \hat{x}_2(n)$. <------> homomorphic deconvolution

The output of the system D_* is called the <u>complex cepstrum</u> of the input signal x(n).

Also, $\hat{y}(n) = \hat{y}_1(n) + \hat{y}_2(n)$ <------> linear filtering of deconvolved signal

and $\quad y(n) = y_1(n) * y_2(n)$ <------> puts output signal in "original form"

<u>How to implement the characteristic system for convolution</u>, D_*:



(a)

<u>First box (applying z-transform or Fourier transform)</u>

Input: $x(n) = x_1(n) * x_2(n)$ Output: $X(z) = X_1(z)X_2(z)$

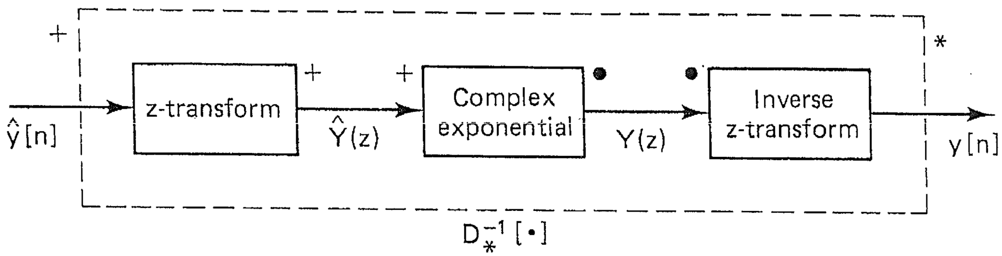
<u>Second box (applying complex logarithm)</u>

Input: $X(z) = X_1(z)X_2(z)$ Output: $\hat{X}(z) = \hat{X}_1(z) + \hat{X}_2(z)$

<u>Third box (applying inverse z-transform)</u>

Input: $\hat{X}(z) = \hat{X}_1(z) + \hat{X}_2(z)$ Output: $\hat{x}(n) = \hat{x}_1(n) + \hat{x}_2(n)$.

How to implement the inverse characteristic system for convolution, D_*^{-1}:



(b)

Input: $\hat{y}(n) = \hat{y}_1(n) + \hat{y}_2(n)$

Output of first box (applying z-transform): $\hat{Y}(z) = \hat{Y}_1(z) + \hat{Y}_2(z)$

Output of second box (applying complex exponentiation):

$$Y(z) = e^{\hat{Y}_1(z)+\hat{Y}_2(z)} = e^{\hat{Y}_1(z)}e^{\hat{Y}_2(z)} = Y_1(z)Y_2(z)$$

Output of third box (applying inverse z-transform): $y(n) = y_1(n) * y_2(n)$

Now consider the linear part of the canonical system for convolution:

Example:

If $\hat{x}_1(n) = 0$ for $n \geq n_0$

and $\hat{x}_2(n) = 0$ for $n < n_0$

then a linear system which can remove $\hat{x}_2(n)$ from the combined signal $\hat{x}_1(n)+\hat{x}_2(n)$, leaving $\hat{x}_1(n)$, is: $\hat{y}(n)=\ell(n)\hat{x}(n)$ where

$$\ell(n)=\begin{cases}1, & n<n_0\\ 0, & n\geq n_0.\end{cases}$$

This is a linear frequency-invariant filter and is often called a "lifter" instead of a filter.
The index n of $\hat{y}(n)$ is called the "quefrency" index.
In the above example, the output of the lifter is $\hat{y}(n)=\hat{x}_1(n)$.
The corresponding frequency domain operation of the lifter system is

$$\hat{Y}(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{X}(e^{j\omega})L(e^{j(\omega-\theta)})d\theta.$$

Section 13.2 Definition of the Complex Cepstrum

Assume that x(n) is a "stable signal" (so that the region of convergence of X(z) includes the unit circle in the z-plane).
Now let $\hat{X}(z)=\log X(z)$ (equation 13.9)
If $\hat{X}(z)$ can be also represented in a power series of the form

$$\hat{X}(z)=\sum_{n=-\infty}^{\infty}\hat{x}(n)z^{-n}$$

which converges for $|z|=1$, then $\hat{x}(n)$ is defined as the complex cepstrum of x(n).

The sequence $\hat{x}(n)$ could be found from $\hat{X}(z) = \log X(z)$ via the inverse z-transform:

$$\hat{x}(n) = \frac{1}{2\pi j} \oint_C \log X(z) z^{n-1} dz$$

where the integration contour C can be the unit circle.

Since the contour C can be chosen as the unit circle in the z-plane, it can also be expressed in terms of the inverse DTFT:

$$\hat{x}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log X(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\log | X(e^{j\omega})) | + j \arg X(e^{j\omega}) \right] e^{j\omega n} d\omega. \qquad \text{(equation 13.13)}$$

Note that the complex log is used, since $X(e^{j\omega})$ can in general be complex.

Note: Despite its name, the "complex cepstrum" is not necessarily complex-valued. In fact, if a signal is real, its complex cepstrum will also be real, as shown below:

If x(n) is real, then $\arg[X(e^{j\omega})]$ is an odd function of ω and $|X(e^{j\omega})|$ is an even function of ω.

Therefore, $\log|X(e^{j\omega})|$ is also an even function. It is also true that $\text{IDTFT}\{\log|X(e^{j\omega})| + j\arg[X(e^{j\omega})]\}$ is real, as shown below:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\log|X(e^{j\omega})|+j\arg[X(e^{j\omega})]\}e^{j\omega n}\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\log|X(e^{j\omega})|+j\arg[X(e^{j\omega})]\}\{\cos\omega n+j\sin\omega n\}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\log|X(e^{j\omega}))|\cos\omega n-\arg[X(e^{j\omega})]\sin\omega n\}d\omega$$

which is real. (The imaginary part of the integral is zero because the integration of odd functions over a symmetric range of values, centered at = 0, is equal to 0.)

Therefore, if x(n) is real (and stable), then the complex cepstrum $\hat{x}(n)$ will also be real (if it exists).

Section 13.3 Properties of the Complex Log

Consider a stable x(n). In order for $\hat{x}(n)$ to exist, it is necessary and sufficient that the region of convergence of $\hat{X}(z)=\log X(z)$ include the unit circle. (This follows from the definition of $\hat{x}(n)$.) If $\hat{x}(n)$ exists, then

$$\hat{X}(e^{j\omega})=\log|X(e^{j\omega})|+j\ \arg[X(e^{j\omega})]. \qquad \text{(equation 13.16)}$$

Both $\log|X(e^{j\omega})|$ and $\arg[X(e^{j\omega})]$ must be continuous functions, since $\hat{X}(z)$ is analytic within its region of convergence, and $\hat{X}(e^{j\omega})$ is equal to $\hat{X}(z)$ on the unit circle.

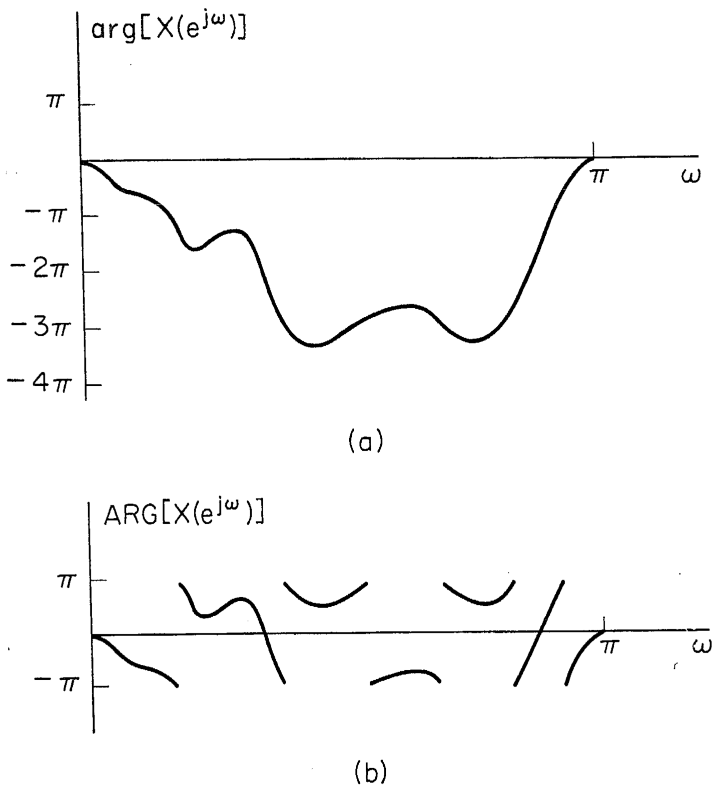
Now consider the evaluation of $\arg[X(e^{j\omega})]$. Most computer routines produce $\text{ARG}[X(e^{j\omega})]$, the "principal value" of the log, which satisfies

$$-\pi \le \text{ARG}[X(e^{j\omega})] \le \pi.$$

The relation between $\text{ARG}[X(e^{j\omega})]$ and the "true phase" $\arg[X(e^{j\omega})]$ is

$$\arg[X(e^{j\omega})] = \text{ARG}[X(e^{j\omega})] + 2\pi r(\omega)$$

where $r(\omega)$ is always integer-valued. Therefore, to find the true $\arg[X(e^{j\omega})]$ it is necessary to "unwrap" $\text{ARG}[X(e^{j\omega})]$. This can be seen from the figure below:



(a) Typical phase curve for a z-transform evaluated on the unit circle. (b) Principal value of the phase in part (a).

If $X(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$, then in order to evaluate the cepstrum we want $\hat{X}(e^{j\omega}) = \hat{X}_1(e^{j\omega}) + \hat{X}_2(e^{j\omega})$.

We will have this relation between $\hat{X}(e^{j\omega})$, $\hat{X}_1(e^{j\omega})$, and $\hat{X}_2(e^{j\omega})$ if $\arg[X(e^{j\omega})]$ is used in evaluating $\log X(e^{j\omega})$. However, this relation will <u>not necessarily</u> be present if $\mathrm{ARG}[X(e^{j\omega})]$ is used in evaluating $\log X(e^{j\omega})$.

<u>Example</u>:

$\arg[X_1(e^{j\omega})] = -100°$ $\mathrm{ARG}[X_1(e^{j\omega})] = -100°$

$\arg[X_2(e^{j\omega})] = -150°$ $\mathrm{ARG}[X_2(e^{j\omega})] = -150°$

$\arg[X(e^{j\omega})] = -250°$ $\mathrm{ARG}[X(e^{j\omega})] = 110°$

<u>Now define the "real cepstrum"</u> as

$$c_x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\log|X(e^{j\omega})|\right]e^{j\omega n}d\omega. \qquad \text{(equation 13.14)}$$

If x(n) is real, $|X(e^{j\omega})|$ and $\log|X(e^{j\omega})|$ are even, and therefore $c_x(n)$ is real.
Since $c_x(n)$ is the DTFT pair with $\mathrm{Re}\{\hat{X}(e^{j\omega})\}$, we know that $c_x(n)$ = "conjugate symmetric" part of $\hat{x}(n)$, that is:

$$c_x(n) = \frac{\hat{x}(n) + \hat{x}^*(-n)}{2}. \qquad \text{(equation 13.15)}$$

Note: $c_x(n)$ is easier to compute than $\hat{x}(n)$, since it does not involve the complex part of the log.
Note: The original signal $x(n)$ cannot be completely recovered from $c_x(n)$.

Calculating the Complex Cepstrum Without Calculating the Complex Log

Method 1:

Consider $\hat{x}(n)$, the complex cepstrum of x(n).

The z-transform of $\hat{x}(n)$ is $\hat{X}(z) = \log X(z)$.

Since $\hat{x}(n)$ is stable, the region of convergence of $\hat{X}(z)$ includes the unit circle.

Therefore, $\hat{X}(z)$ is analytic on the unit circle. This means that $\hat{X}(z)$ and all its derivatives are continuous on the unit circle.

Now consider the derivative of $\hat{X}(z)$:

$$\hat{X}'(z) = \frac{d}{dz}\log X(z) = \frac{1}{X(z)}\frac{dX(z)}{dx} = \frac{X'(z)}{X(z)} \qquad \text{(equation 13.21)}$$

Using the z-transform property which states that

$$-z\frac{d\hat{X}(z)}{dz} = \text{"z – transform" of } [n\hat{x}(n)],$$

we can obtain the following relation:

$$-n\hat{x}(n) = \frac{1}{2\pi j}\oint_C \frac{zX'(z)}{X(z)} z^{n-1} dz$$

where the contour of integration C can be the unit circle.

For $n \neq 0$, $\hat{x}(n) = -\frac{1}{2\pi jn}\oint_C \frac{zX'(z)}{X(z)} z^{n-1} dz.$

For n=0 , we can find $\hat{x}(0)$ using

$$\hat{x}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}(e^{j\omega})\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|X(e^{j\omega})|\,d\omega + j\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \arg[X(e^{j\omega})]\,d\omega}_{0}.$$

<u>Method 2</u>

Since $\hat{X}'(z) = \frac{X'(z)}{X(z)}$, we can write

$$z\hat{X}'(z)X(z) = zX'(z).$$

The <u>inverse z-transform of the left hand side</u> can be expressed as $[-n\hat{x}(n)]*x(n)$.

The <u>inverse z-transform of the right hand side</u> can be expressed as **$-nx(n)$**. .

Therefore, we can write x(n) as

$$x(n) = \sum_{k=-\infty}^{\infty}\left(\frac{k}{n}\right)\hat{x}(k)x(n-k), \quad n \neq 0. \qquad \text{(equation 13.26)}$$

Later, we will show that by imposing additional constraints, this above can be rearranged to solve for $\hat{x}(n)$ in terms of x(n).

Complex Cepstrum for Exponential Signals

Consider the special case where $X(z)$ can be expressed in the form:

$$X(z) = \frac{Az^r \prod_{k=1}^{M_i}(1-a_k z^{-1}) \prod_{k=1}^{M_o}(1-b_k z)}{\prod_{k=1}^{N_i}(1-c_k z^{-1}) \prod_{k=1}^{N_o}(1-d_k z)} \qquad \text{(equation 13.29)}$$

where $|a_k|, |b_k|, |c_k|, \text{and } |d_k| < 1$. Note that the a_k and c_k correspond to roots inside the unit circle while b_k and d_k represent roots outside the unit circle.

Then

$$\hat{X}(z) = \log(A) + \log(z^r) + \sum_{k=1}^{M_i} \log(1-a_k z^{-1}) + \sum_{k=1}^{M_o} \log(1-b_k z) - \sum_{k=1}^{N_i} \log(1-c_k z^{-1}) - \sum_{k=1}^{N_o} \log(1-d_k z).$$

(equation 13.30)

Now consider the case where r = 0 and $A = |A|$. (We can obtain this condition by shifting x(n) and multiplying by -1, if necessary.)

Therefore, $\log A = \log|A|$ and $\log(z^r) = 0$.

Now consider the other terms of $\hat{X}(z) = \log X(z)$:

$$\log(1-\alpha z^{-1}) = -\sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}, \qquad \text{for } |\alpha z^{-1}| < 1, \text{ or } |z| > \alpha$$ (equation 13.34)

and

$$\log(1-\beta z) = -\sum_{n=1}^{\infty} \frac{\beta^n}{n} z^{n}, \qquad \text{for } |\beta z| < 1, \text{ or } |z| < \frac{1}{\beta}.$$ (equation 13.35)

Therefore,

$$\hat{X}(z) = \log|A| + \sum_{k=1}^{M_i}\left(-\sum_{n=1}^{\infty}\frac{a_k^n}{n}z^{-n}\right) + \sum_{k=1}^{M_o}\left(-\sum_{n=1}^{\infty}\frac{b_k^n}{n}z^{n}\right) - \sum_{k=1}^{N_i}\left(-\sum_{n=1}^{\infty}\frac{c_k^n}{n}z^{-n}\right) - \sum_{k=1}^{N_o}\left(-\sum_{n=1}^{\infty}\frac{d_k^n}{n}z^{n}\right)$$

$$= \log|A| + \sum_{n=1}^{\infty}\left(-\sum_{k=1}^{M_i}\frac{a_k^n}{n} + \sum_{k=1}^{N_i}\frac{c_k^n}{n}\right)z^{-n} + \sum_{n=1}^{\infty}\left(-\sum_{k=1}^{M_o}\frac{b_k^n}{n} + \sum_{k=1}^{N_o}\frac{d_k^n}{n}\right)z^{n}.$$

In the second outer summation, replace n with -n and adjust limits accordingly:

$$= \log|A| + \sum_{n=1}^{\infty}\left(-\sum_{k=1}^{M_i}\frac{a_k^n}{n} + \sum_{k=1}^{N_i}\frac{c_k^n}{n}\right)z^{-n} + \sum_{n=-1}^{-\infty}\left(\sum_{k=1}^{M_o}\frac{b_k^{-n}}{n} - \sum_{k=1}^{N_o}\frac{d_k^{-n}}{n}\right)z^{-n}.$$

By comparing the above expression for $\hat{X}(z)$ with the general expression for the z-transform of $\hat{x}(n)$ that is, $\hat{X}(z) = \sum_{n=-\infty}^{\infty} \hat{x}(n) z^{-n}$, we see that $\hat{x}(n)$ can be expressed as:

$$\hat{x}(n) = \begin{cases} \log|A|, & n = 0 \\ -\sum_{k=1}^{M_i} \frac{a_k^n}{n} + \sum_{k=1}^{N_i} \frac{c_k^n}{n}, & n > 0 \\ \sum_{k=1}^{M_o} \frac{b_k^{-n}}{n} - \sum_{k=1}^{N_o} \frac{d_k^{-n}}{n}, & n < 0 \end{cases}$$

(equation 13.36 (a))

(equation 13.36 (b))

(equation 13.36 (c))

Call the above approach Method 3 for finding $\hat{x}(n)$.