ECE 8440 UNIT 23

Section 13.6.5 The Use of Exponential Weighting

Exponential weighting of a sequence x(n) is defined by

$$w(n) = \alpha^n x(n). \qquad \text{(equation 13.69)}$$

Exponential weighting can be used to avoid or lessen some of the problems involved with evaluating the complex cepstrum $\hat{x}(n)$. Note that the z-transform of $w(n)$ is

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{\alpha}\right)^{-n} = X\left(\frac{z}{\alpha}\right). \qquad \text{(equation 13.70)}$$

If the Region of Convergence of X(z) is

$$r_R < |z| < r_L$$

then the Region of Convergence of W(z) is

$r_R < \left|\frac{z}{\alpha}\right| < r_L$, which can be also expressed as

$$|\alpha| r_R < |z| < |\alpha| r_L.$$

If X(z) has a pole or zero at $z = z_0 = r_0 e^{j\theta_0}$, then W(z) has a pole of zero when $\frac{z}{\alpha} = r_0 e^{j\theta_0}$, which corresponds to $z = \alpha r_0 e^{j\theta_0}$, which has a radius of αr_0 and the same angle as the pole or zero of X(z).

Note that <u>the operation of exponential weighting commutes with the operation of convolution</u>, as shown below:

Assume that $x(n) = x_1(n) * x_2(n)$ and let $w(n) = \alpha^n x(n)$.

Then $W(z) = X\left(\frac{z}{\alpha}\right) = X_1\left(\frac{z}{\alpha}\right) X_2\left(\frac{z}{\alpha}\right)$. (equation 13.71)

Therefore,

$$w(n) = \mathrm{IDTFT}\left[X_1\left(\frac{z}{\alpha}\right)\right] * \mathrm{IDTFT}\left[X_2\left(\frac{z}{\alpha}\right)\right]$$

$$= \alpha^n x_1(n) * \alpha^n x_2(n)$$

$$= w_1(n) * w_2(n) \qquad \text{(equation 13.72)}$$

where $w_1(n) = \alpha^n x_1(n)$ and $w_2(n) = \alpha^n x_2(n)$.

The frequency domain version of this is

$W(z) = W_1(z) W_2(z)$.

Therefore, in computing the complex cepstrum of w(n), which is the exponentially weighted version of x(n), we have

$\hat{W}(z) = \log[W(z)] = \log[W_1(z)] + \log[W_2(z)]$ and $\hat{w}(n) = \hat{w}_1(n) + \hat{w}_2(n)$. (equation 13.73)

Applications of exponential weighting

1. Moving poles and zeros off the unit circle, so that complex cepstrum can be calculated.

Because of the way the complex cepstrum is defined, it can exist only if the Region of Convergence of X(z) and also of $\hat{X}(z) = \log[X(z)]$ include the unit circle in the z-plane.

Poles of X(z) are also poles of $\hat{X}(z) = \log[X(z)]$.

Zeros of X(z) become poles of $\hat{X}(z)$ because of the log operation. Therefore, if X(z) has any poles or zeros on the unit circle, the complex cepstrum, as we have defined it, cannot exist.

The above situation can be remedied by exponential weighting, using $\alpha < 1$, to move any poles and zeros which are on the unit circle to new locations inside the unit circle (keeping the angles of the poles and zeros unchanged.)

2. Converting a non-minimum phase signal to a signal that is minimum-phase.

If z_{max} is the largest magnitude of any of the poles and zeros of X(z) where $z_{max} > 1$, we can generate a minimum-phase version of x(n) by using exponential weighting,

$$w(n) = \alpha^n x(n)$$

where α is chosen to satisfy $| z_{max} \alpha | < 1$.

3. Computing the complex cepstrum without computing the complex log (which would involve phase unwrapping)

The exponentially weighted signal $w(n) = \alpha^n x(n)$ has z-transform $W(z) = X\left(\frac{z}{\alpha}\right)$. If the Region of Convergence of X(z) is

$r_R < |z| < r_L$,

then the Region of Convergence of W(z) is $|\alpha| r_R < |z| < |\alpha| r_L$,

and the poles and zeros of X(z) are shifted radially by a factor of α.

If X(z) has no poles or zeros on the unit circle then α can be chosen so that no poles or zeros of X(z) move across the unit circle in forming W(z). Then, the Region of Convergence of W(z) will also include the unit circle. (This is a necessary condition for its complex cepstrum to exit.)

Note that the poles of $\hat{W}(z) = \log[W(z)]$ consist of the poles <u>and</u> zeros of W(z) and the Region of Convergence of $\hat{W}(z)$ is given by

$A < |z| < B$

where A is the largest magnitude of the poles of $\hat{W}(z)$ which are inside the unit circle, and B is the smallest magnitude of the poles of $\hat{W}(z)$ which are outside the unit circle.

Therefore, the Region of Convergence of $\hat{W}(z)$ includes the unit circle, which is another necessary condition for the complex cepstrum $\hat{w}(n)$ to exist.

Since $\hat{W}(z) = \log W(z) = \log X\left(\frac{z}{\alpha}\right)$, the complex cepstrum of w(n) is

$$\hat{w}(n) = \text{IDTFT}\left[\log X\left(\frac{z}{\alpha}\right)\right] = \alpha^n\ \text{IDTFT}\left[\log X(z)\right] = \alpha^n \hat{x}(n).$$

Now assume that instead of the complex cepstrum, we have calculated the real cepstrum $c_x(n)$. For the case where x(n) is real, $\hat{x}(n), w(n)$, and $\hat{w}(n)$ will also be real. In this case, $c_x(n)$ is the even part of $\hat{x}(n)$ and $c_w(n)$ is the even part of $\hat{w}(n)$. These relations can be expressed as

$$c_x(n) = \frac{\hat{x}(n) + \hat{x}(-n)}{2} \quad \text{and} \qquad \text{equation *}$$

$$c_w(n) = \frac{\hat{w}(n) + \hat{w}(-n)}{2} = \frac{\alpha^n \hat{x}(n) + \alpha^{-n}\hat{x}(-n)}{2} \quad . \qquad \text{equation **}$$

From equation * we can write

$$\hat{x}(-n) = 2c_x(n) - \hat{x}(n)$$

Substituting the above expression for $\hat{x}(-n)$ into equation ** gives:

$$c_w(n) = \frac{\alpha^n \hat{x}(n) + \alpha^{-n}[2c_x(n) - \hat{x}(n)]}{2}$$

$$2c_w(n) = \alpha^n \hat{x}(n) + \alpha^{-n}[2c_x(n) - \hat{x}(n)]$$

$$2c_w(n) - \alpha^{-n} 2c_x(n) = \hat{x}(n)[\alpha^n - \alpha^{-n}]$$

$$\hat{x}(n) = \frac{2c_w(n) - \alpha^{-n}2c_x(n)}{[\alpha^n - \alpha^{-n}]}$$

$$= \frac{2\alpha^n c_w(n) - 2c_x(n)}{[\alpha^{2n} - 1]}$$

$$\hat{x}(n) = \frac{2[c_x(n) - \alpha^n c_w(n)]}{1 - \alpha^{2n}}, \quad n \neq 0. \qquad \text{equation ***}$$

Possible difficulty: If X(z) has a pole or zero very close to the unit circle, then it may be necessary for α to be close to 1. This can cause problems in evaluating the equation *** since its denominator would be very small in this case.

Section 13.9 – The Complex Cepstrum for a Simple Multipath Model

Consider the general case a signal x(n) which consists of the convolution of two component signals v(n) and p(n):

$x(n) = v(n)*p(n).$ (equation 13.92a)

In the z-domain this relation is

$X(z) = V(z)P(z).$ (equation 13.92b)

Consider the signal p(n) to have the form:

$p(n) = \delta(n) + \beta\delta(n-N_0)+\beta^2\delta(n-2N_0).$ (equation 13.93a)

The z-transform of p(n) is

$$P(z) = 1 + \beta z^{-N_0} + \beta^2 z^{-2N_0}.$$

By using the formula for a finite geometric series, P(z) can be expressed as

$$P(z) = \frac{1-\beta^3 z^{-3N_0}}{1-\beta z^{-N_0}}. \qquad \text{(equation 13.93b)}$$

The <u>denominator</u> is equal to zero when $\beta z^{-N_0} = 1$ or $z^{N_0} = \beta$.

Therefore, <u>roots of the denominator</u> are at

$$z_k = \beta^{\frac{1}{N_0}} e^{j2\pi k/N_0}, \qquad k = 0, 1, \ldots, N_0 - 1.$$

The <u>numerator</u> is equal to zero when

$\beta^3 z^{-3N_0} = 1$ or $\beta^3 = z^{3N_0}$..

<u>Roots of the numerator</u> are therefore at

$$z_k = \left(\beta^3\right)^{\left(\frac{1}{3N_0}\right)} e^{j2\pi k/(3N_0)} = \beta^{\frac{1}{N_0}} e^{j2\pi k/(3N_0)}, \qquad k = 0, 1, \ldots, 3N_0 - 1.$$

As would be expected for finite length signal, all the roots of the denominator of its z-transform are "cancelled" by roots of the numerator. (2/3 of the roots of the numerator remain "uncancelled.")

Now assume that the signal v(n) is the unit sample response of a second order system V(z) where

$$V(z) = \frac{b_0 + b_1 z^{-1}}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}. \qquad \text{(equation 13.94a)}$$

The time domain representation of this system is

$$v(n) = b_0 w(n) + b_1 w(n-1) \text{ where} \qquad \text{(equation 13.94b)}$$

$$w(n) = \frac{r^n}{4\sin^2\theta}\{\cos(\theta n) - \cos[\theta(n+2)]\}u(n), \qquad \theta \neq 0, \pi \qquad \text{(equation 13.94c)}$$

The figure below provides a plot of the poles and zeros of X(z)=P(z)V(z) for the following set of parameter values:

$b_0 = .98$
$b_1 = 1$
$\beta = r = 0.9$
$\theta = \pi/6$
$N_0 = 15$

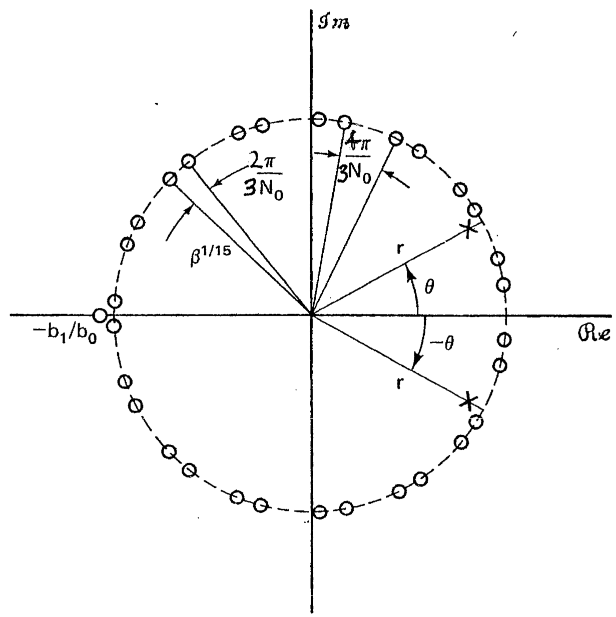


Figure 13.9 Pole-zero plot of the z-transform *X(z) = V(z) P(z)* for the example of Figure 13.10.

The figure below shows plots of v(n), p(n). and x(n) corresponding to the poles and zeros in the previous figure.

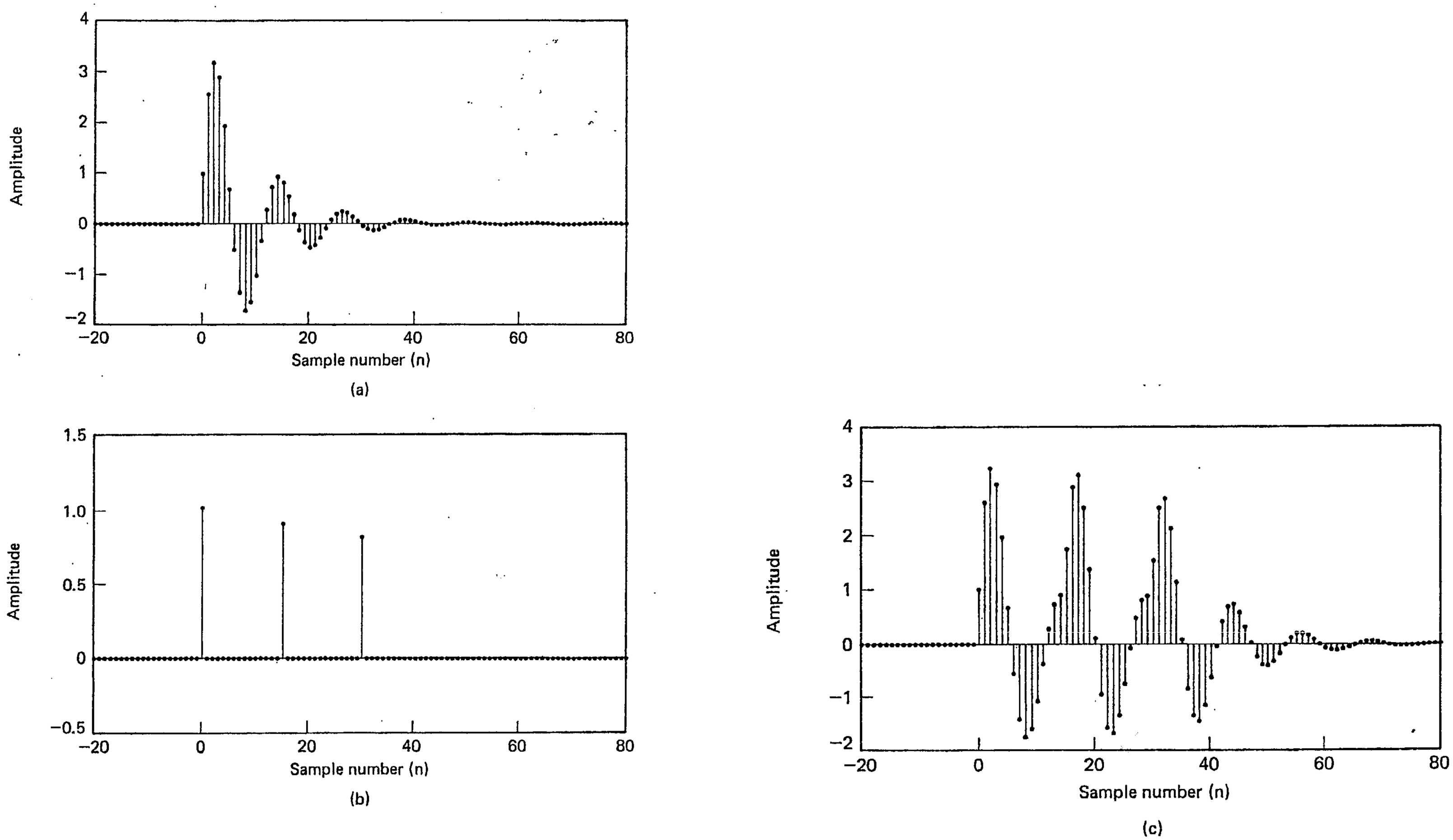


Figure 13.10 The sequences (a) *v[n]*, (b*) p[n*,. and (c) *x(n)* corresponding to the pole-zero plot of Figure 13.9.

The signal model described above can serve as a basis for applying cepstral analysis to a number of different application areas. For example:

Speech Processing

v(n): combined effect of glottal pulse shape and resonant effect of vocal tract

p(n): periodic excitation signal from the vocal cords

Seismic Data Analysis

v(n): waveform of acoustic pulse propagating in the earth do to a dynamite explosion or due to an earthquake

p(n): a sequence that represents reflections at boundaries between underground layers having different propagation characteristics

Communication Systems:

v(n): signal transmitted over a multi-path channel

p(n): unit sample response of the multi-path channel

To demonstrate the above signal model represents a situation where cepstral analysis can be used effectively to perform deconvolution of signal component, we now calculate the complex cepstrum for previous example:

Begin by writing V(z) in the form we used for the "polynomial rooting" method for rational system (refer to equation 13.32):

$$V(z) = \frac{b_1 z^{-1}[1+(b_0/b_1)z]}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}. \qquad \text{(equation 13.97)}$$

As discussed earlier, the z^{-1} term corresponds to a simple time shift, and is usually discarded for computational reasons. (Including this term would cause discontinuities of $\arg[x(e^{j\omega})]$ at $\omega = \pm\pi$, and therefore $\hat{V}(z)$ would not be analytic on the unit circle, which is required, due to the definition of the complex cepstrum.) Therefore, we proceed with representing $\hat{V}(z)$ as follows:

$$V(z) = \frac{b_1[1+(b_0/b_1)z]}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}. \qquad \text{(equation 13.98)}$$

Note that the two poles are <u>inside the unit circle</u> and the zero is at $-(b_1/b_0)$, which is <u>outside the unit circle</u>, since $b_0 = .98$ and $b_1 = 1$ for this example. Therefore, using the expression of equation 13.36, we can write

$$\hat{v}[0] = \log[b_1]$$ (equation 13.99a)

$$\hat{v}[n] = \frac{1}{n}\left[(re^{j\theta})^n + (re^{-j\theta})^n\right], \quad n > 0$$ due to the two poles <u>inside</u> unit circle (equation 13.99b)

$$\hat{v}[n] = \frac{1}{n}\left(\frac{-b_0}{b_1}\right)^{-n}, \quad n < 0.$$ due to the zero <u>outside</u> unit circle (equation 13.99c)

Note that as expected, $\hat{v}[n]$ decreases as $1/n$.

For future reference, note that we can write $\hat{v}[n]$ as

$$\hat{v}[n] = \hat{v}_a[n] + \hat{v}_b[n] + \hat{v}_c[n]$$

where

$$\hat{v}_a[n] = \log[b_1], \quad n = 0$$

$$\hat{v}_b[n] = \frac{1}{n}\left[(re^{j\theta})^n + (re^{-j\theta})^n\right], \quad n > 0$$

$$= \frac{r^n}{n}\left[2\cos(\theta n)\right], \quad n > 0$$

$$\hat{v}_c[n] = \frac{1}{n}\left(\frac{-b_0}{b_1}\right)^{-n}, \quad n < 0.$$

Now determine the cepstrum of p[n], the other component of x[n]:

$$\hat{P}(z) = \log P(z) = \log(1-\beta^3 z^{-3N_0}) - \log(1-\beta z^{-N_0}).$$

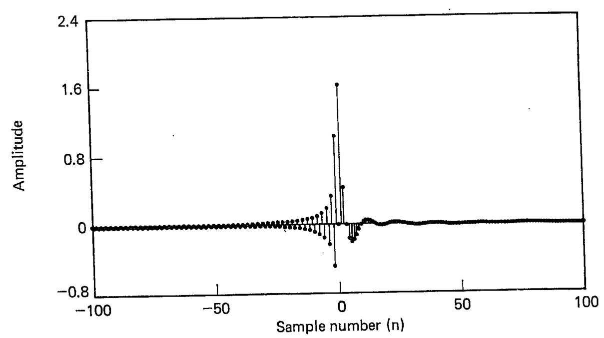
Now use the series expansion for the log function to represent $\hat{P}(z)$ as

$$\hat{P}(z) = -\sum_{k=1}^{\infty} \frac{\beta^{3k}}{k} z^{-3N_0 k} + \sum_{k=1}^{\infty} \frac{\beta^{k}}{k} z^{-N_0 k}. \qquad \text{(equation 13.101)}$$

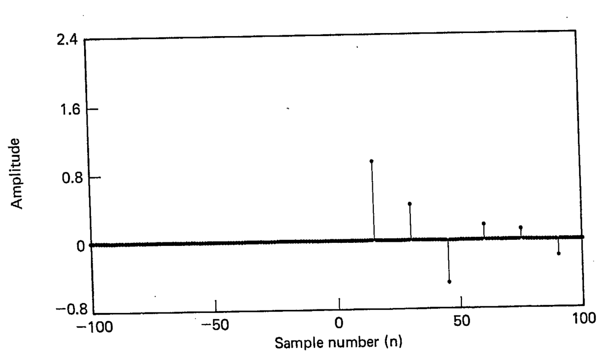
Using the delay property of z-transforms, we can write the expression for $\hat{p}[n]$ as

$$\hat{p}(n) = -\sum_{k=1}^{\infty} \frac{\beta^{3k}}{k} \delta(n-3N_0 k) + \sum_{k=1}^{\infty} \frac{\beta^{k}}{k} \delta(n-N_0 k). \qquad \text{(equation 13.102)}$$

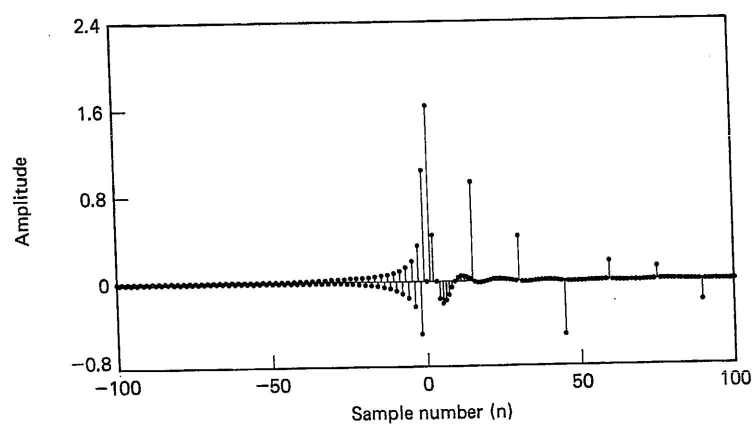
Note that $\hat{p}[n]$ is equal to 0 except when k is a positive integer multiple of N_0.
Plots of $\hat{v}[n]$, $\hat{p}[n]$, and $\hat{x}(n) = \hat{v}[n] + \hat{p}(n)$ are shown in the following figure:



(a)



(b)



(c)

Figure 13.11 The sequences (a) $\hat{v}[n]$, (b) $\hat{p}[n]$, and (c) $\hat{x}[n]$.

The real cepstrum of x(n), $c_x(n)$, is the even part of $\hat{x}[n]$ and can therefore be expressed as

$$c_x(n) = \frac{1}{2}[\hat{x}(n) + \hat{x}(-n)].$$

equation 13.104

We also know that

$$c_x(n) = c_v(n) + c_p(n).$$

equation 13.105

First, focus on $c_v(n)$. Since $c_v(n)$ is the even part of $\hat{v}[n]$, it is related to $\hat{v}[n]$ via

$$c_v(n) = \frac{1}{2}[\hat{v}[n] + \hat{v}[-n]].$$

Recall that we can express $\hat{v}[n]$ as

$\hat{v}[n] = \hat{v}_a[n] + \hat{v}_b[n] + \hat{v}_c[n]$ where

$$\hat{v}_a[n] = \log[b_1], \qquad n = 0$$

$$\hat{v}_b[n] = \frac{r^n}{n}\left[2\cos(\theta n)\right], \qquad n > 0$$

$$\hat{v}_c[n] = \frac{1}{n}\left(\frac{-b_0}{b_1}\right)^{-n}, \qquad n < 0.$$

Therefore, for n = 0 we can express $c_v(n)$ as

$$c_v(0) = \frac{1}{2}[\hat{v}_a(0) + \hat{v}_a(0)] = \frac{1}{2}[\log[b_1] + \log[b_1]] = \log[b_1].$$

For $n > 0$, we can write $c_v(n)$ as

$$c_v(n) = \frac{1}{2}[\hat{v}_b(n) + \hat{v}_c(-n)].$$

$$= \frac{r^n}{n}\left[\cos(\theta n)\right] + \frac{1}{2(-n)}\left(\frac{-b_0}{b_1}\right)^n, \qquad n > 0$$

$$= \frac{r^n}{n}\left[\cos(\theta n)\right] - \frac{1}{2n}(-1)^n\left(\frac{b_0}{b_1}\right)^n, \qquad n > 0.$$

For n < 0, we can write $c_v(n)$ as

$$c_v(n) = \frac{1}{2}[\hat{v}_b(-n) + \hat{v}_c(n)]$$

$$= \frac{r^{-n}}{(-n)}\left[\cos(\theta n)\right] + \frac{1}{2n}(-1)^{-n}\left(\frac{b_0}{b_1}\right)^{-n}, \qquad n < 0.$$

Therefore, for all $n \neq 0$ we can write

$$c_v(n) = \frac{r^{|n|}}{|n|}\left[\cos(\theta n)\right] - \frac{1}{2|n|}(-1)^n\left(\frac{b_0}{b_1}\right)^{|n|}, \qquad n \neq 0.$$

One way to represent the above expression is

$$c_v(n) = \sum_{k=1}^{\infty} r^k\left[\frac{\cos(\theta k)}{k}\right][\delta(n-k) + \delta(n+k)] - \sum_{k=1}^{\infty}\frac{1}{2k}(-1)^k(b_0/b_1)^k[\delta(n-k) + \delta(n+k)], \qquad n \neq 0.$$

An expression of this form that is valid for all n is

$$c_v(n) = \log(b_1)\delta(n) + \sum_{k=1}^{\infty} r^k\left[\frac{\cos(\theta k)}{k}\right][\delta(n-k) + \delta(n+k)] - \sum_{k=1}^{\infty}\frac{1}{2k}(-1)^k(b_0/b_1)^k[\delta(n-k) + \delta(n+k)].$$

(see equation 13.106a)

We can also use equation 13.102 to express $c_p(n)$ as

$$c_p(n) = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{\beta^{3k}}{k}\right)\left[\delta(n-3N_0k)+\delta(n+3N_0k)\right]+\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{\beta^{k}}{k}\right)\left[\delta(n-N_0k)+\delta(n+N_0k)\right].$$

(equation 13.106b)

The following figure shows the sequences $c_v(n)$, $c_p(n)$, and $c_x(n)$ of the above example.

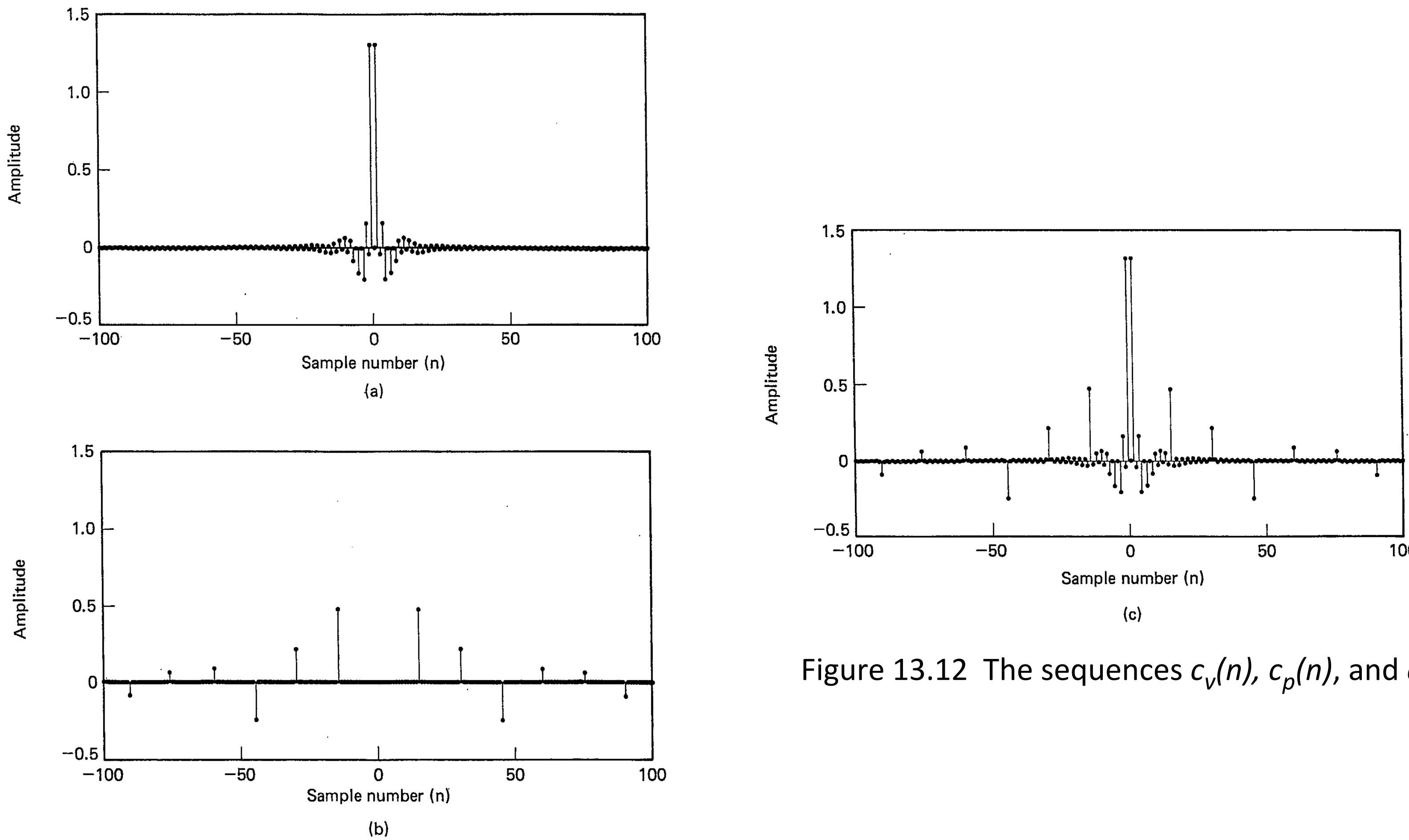


Figure 13.12 The sequences $c_v(n)$, $c_p(n)$, and $c_x(n)$

Using the DFT to Compute the Complex Cepstrum of the Signal x(n) of Figure 13. 10 (c)

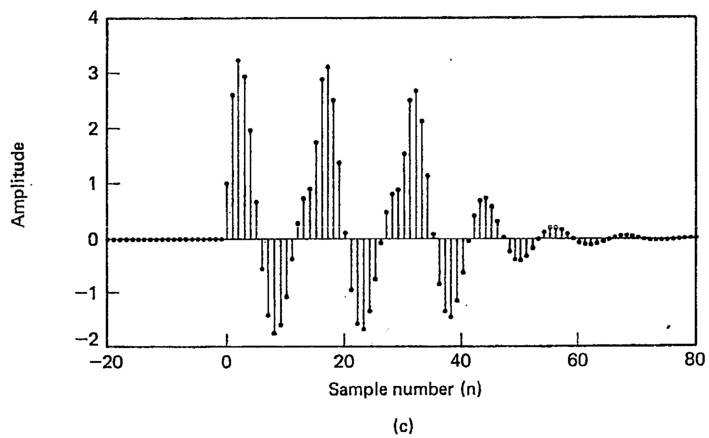
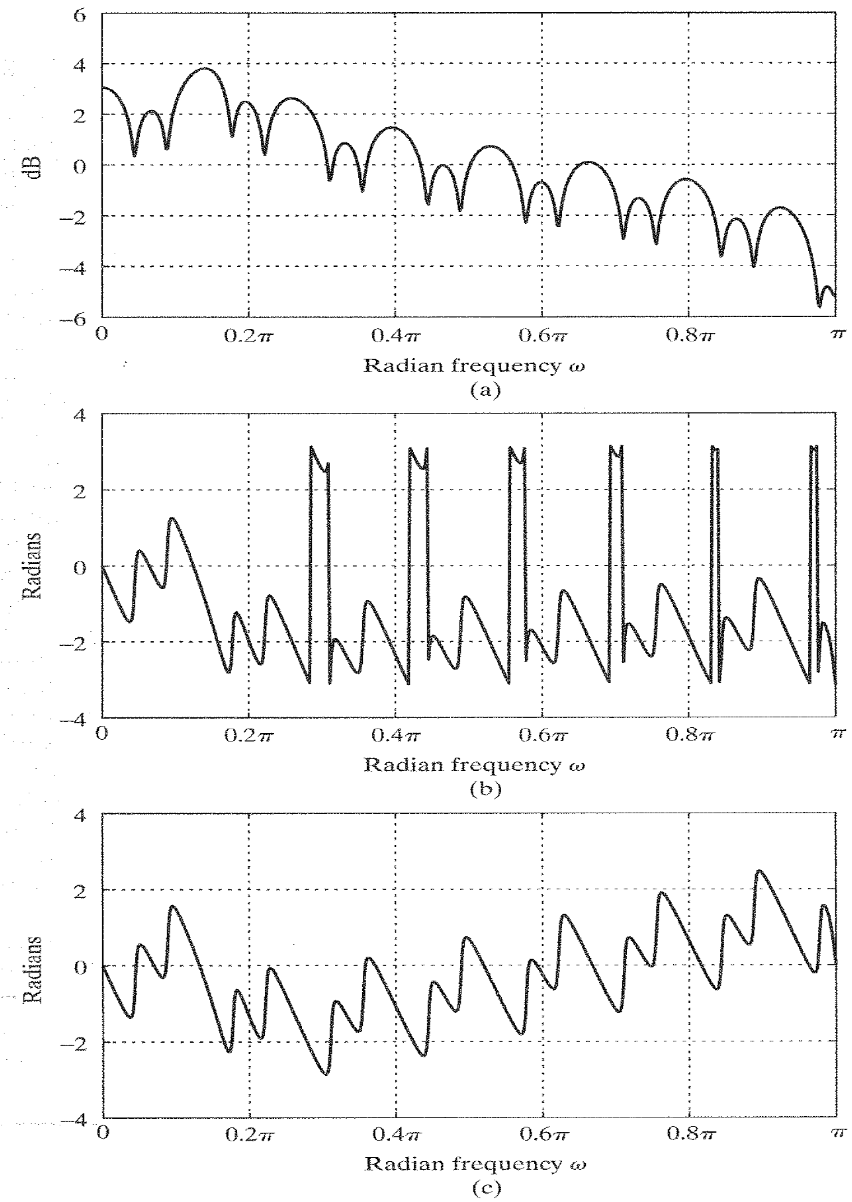


Figure 13.10c The sequence x(n) corresponding to the pole-zero plot of Fig. 13.9

- This (the DFT-based implementation) will be an aliased version of the ceptrum obtained by an analytic evaluation of the complex cepstrum of x(n), which was based on the form of X(z).

- DFT length of 1024 will be used.

- First, look at the output of the second step in the system D_*
(The result after applying the DFT, then applying the log operation.)



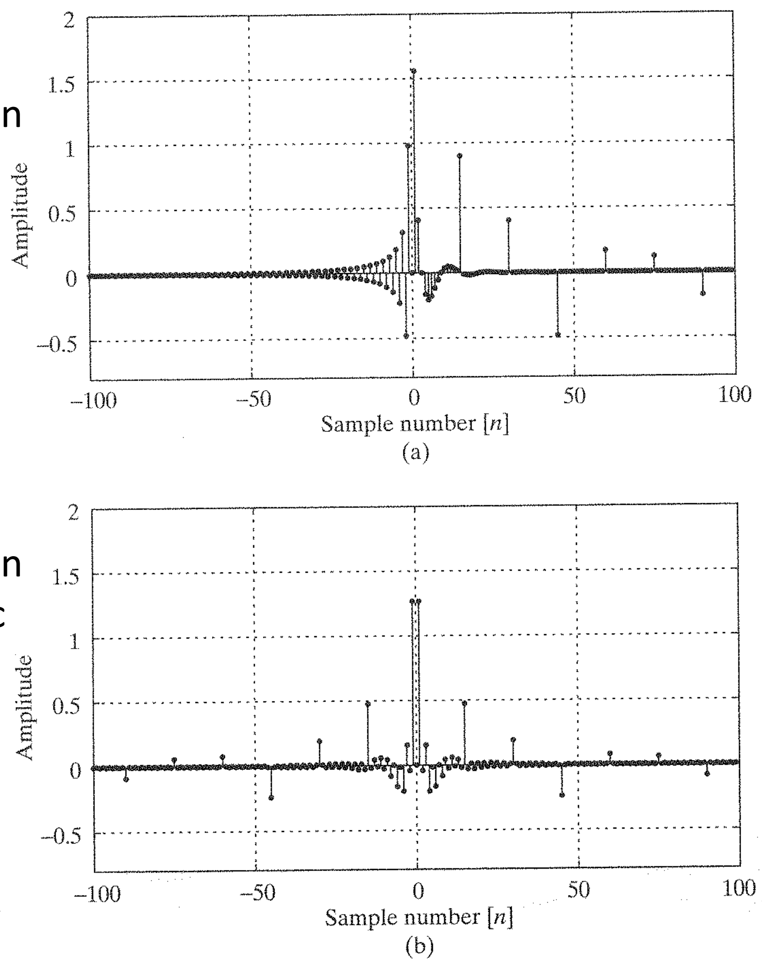
Note the general appearance of a rapidly varying periodic component added to a more slowly varying component.

Figure 13.13 Fourier transforms of $x[n]$ in Figure 13.10. (a) Log magnitude. (b) Principal value of the phase. (c) Continuous "unwrapped" phase after removing a linear-phase component from part (b). The DFT samples are connected by straight lines.

Now look at the output of step 3 (the inverse DFT) in system D_* ,
which is the complex cepstrum:

aliased version of Fig. 13.11c

aliased version of Fig. 13.12c



- Pulses at intervals of 15 are due to periodic pulse train of p(n)
- Other component of complex cepstrum, $\hat{x}_p(n)$ decreases as 1/n.
- Since x(n) has a zero outside the unit circle (and was therefore not a minimum phase signal), $\hat{x}_p(n)$ is not 0 for n < 0.

Figure 13.14 (a) Complex cepstrum $\hat{x}_p[n]$ of sequence in Figure 13.10(c). (b) Cepstrum $c_x[n]$ of sequence in Figure 13.10(c).

We can use a linear "lilter" to extract either the "low-time" part of the complex cepstrum or the "high-time" part.

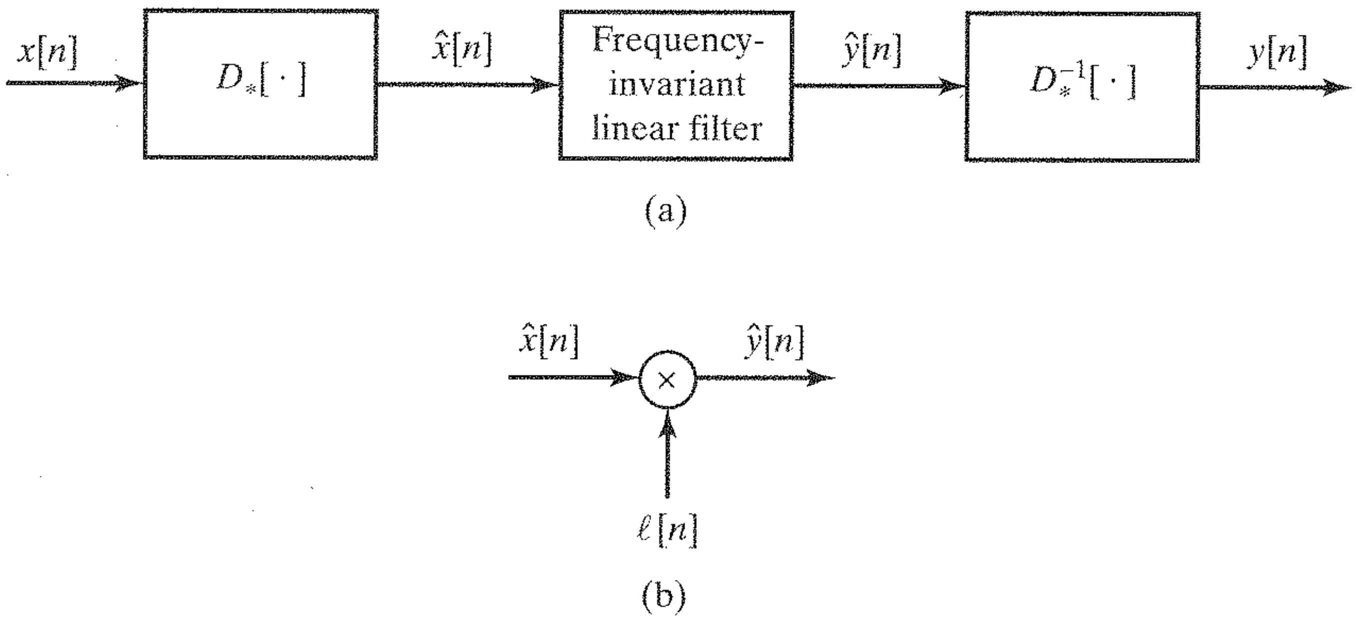


Figure 13.15 (a) System for homomorphic deconvolution. (b) Time-domain representation of frequency-invariant filtering.

For the DFT, there are no negative indices. Therefore, the "low-time" and "high-time" parts of the complex cepstrum are obtained by extracting from the region $0 \leq n \leq N\text{-}1$ the appropriate segments from the periodic extension (with period N) of the desired "lifter."

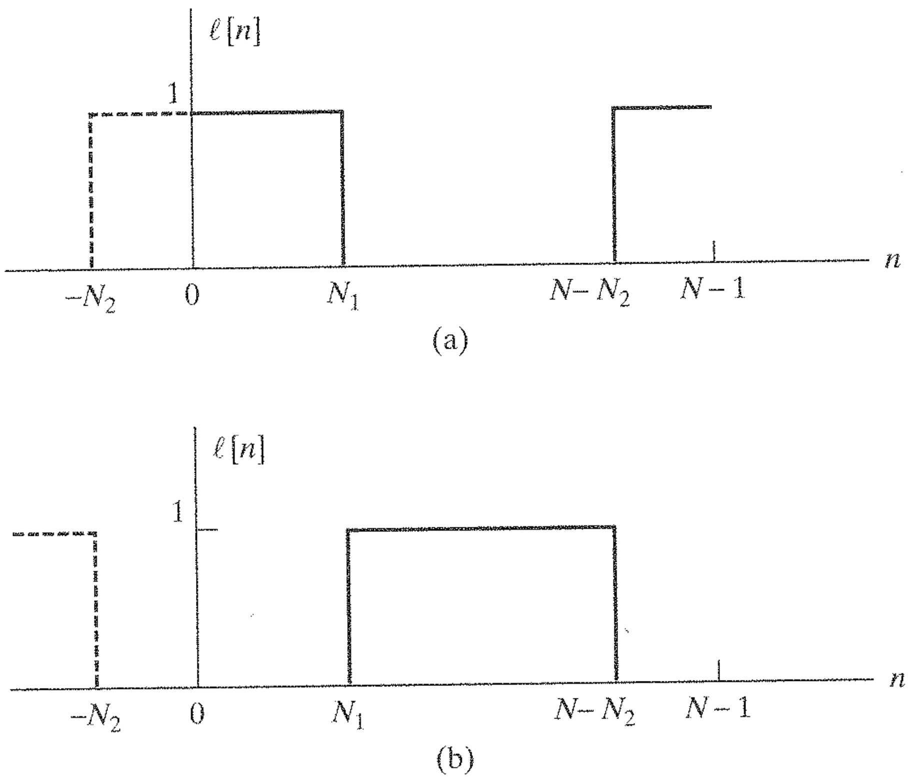


Figure 13.16 Time response of frequency-invariant linear systems for homomorphic deconvolution. (a) Lowpass system. (b) Highpass system. (Solid line indicates envelope of the sequence $\ell[n]$ as it would be applied in a DFT implementation. The dashed line indicates the periodic extension.)

After extracting the low-time part of the complex ceptrum, the signal component v(n) can be approximately reproduced using a DFT-implementation of system D_*^{-1}

(DFT, then exponentiation, then IDFT)

The dashed lines in parts a and b are the output of the DFT step

(Compare this with the signal v(n) in Fig 13.10(a)

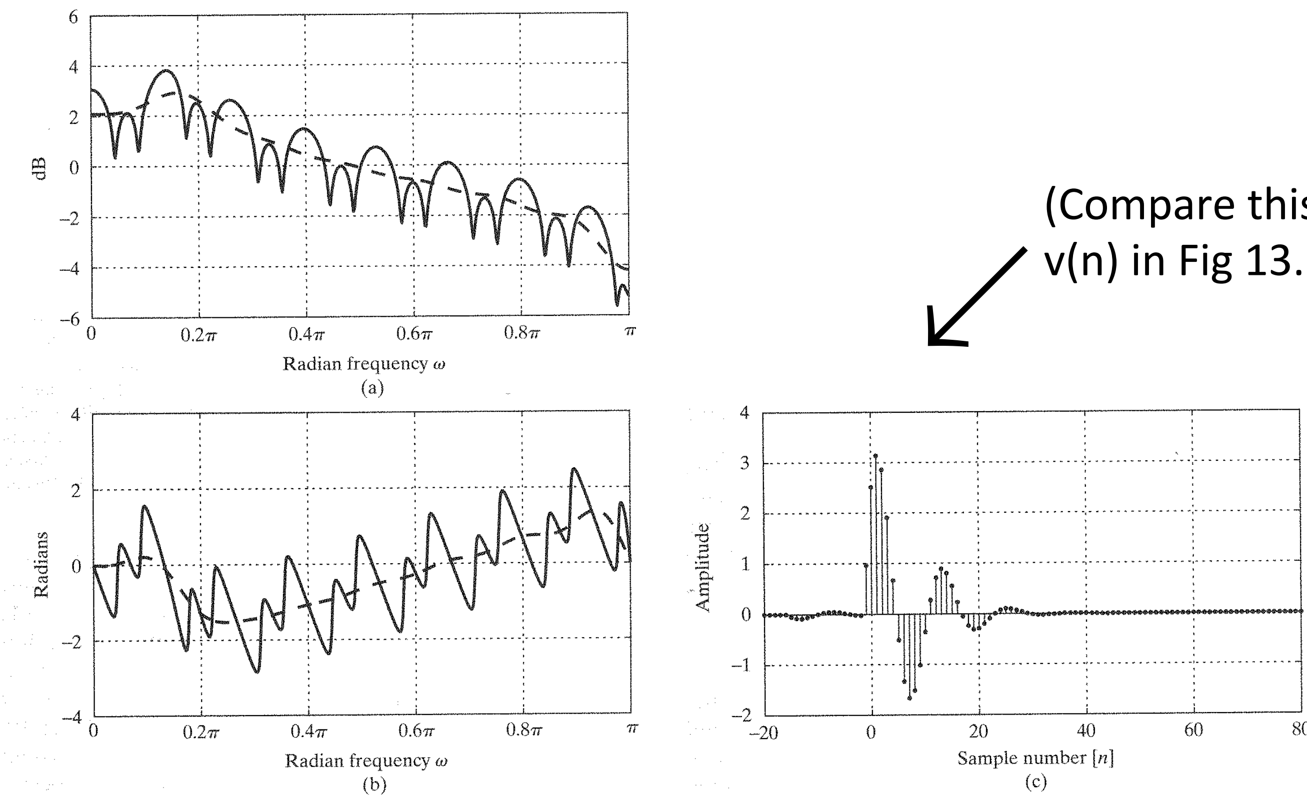


Figure 13.17 Lowpass frequency-invariant linear filtering in the system of Figure 13.15. (a) Real parts of the Fourier transforms of the input (solid line) and output (dashed line) of the lowpass system with $N_1 = 14$ and $N_2 = 14$ in Figure 13.16(a). (b) Imaginary parts of the input (solid line) and output (dashed line). (c) Output sequence $y[n]$ for the input of Figure 13.10(c).

After extracting the high-time part of the complex ceptrum, the signal component p(n) can be approximately produced using a DFT- implementation of system.

(DFT, then exponentiation, the IDFT)

(Compare this with the signal p(n) in Fig 13.10(b)

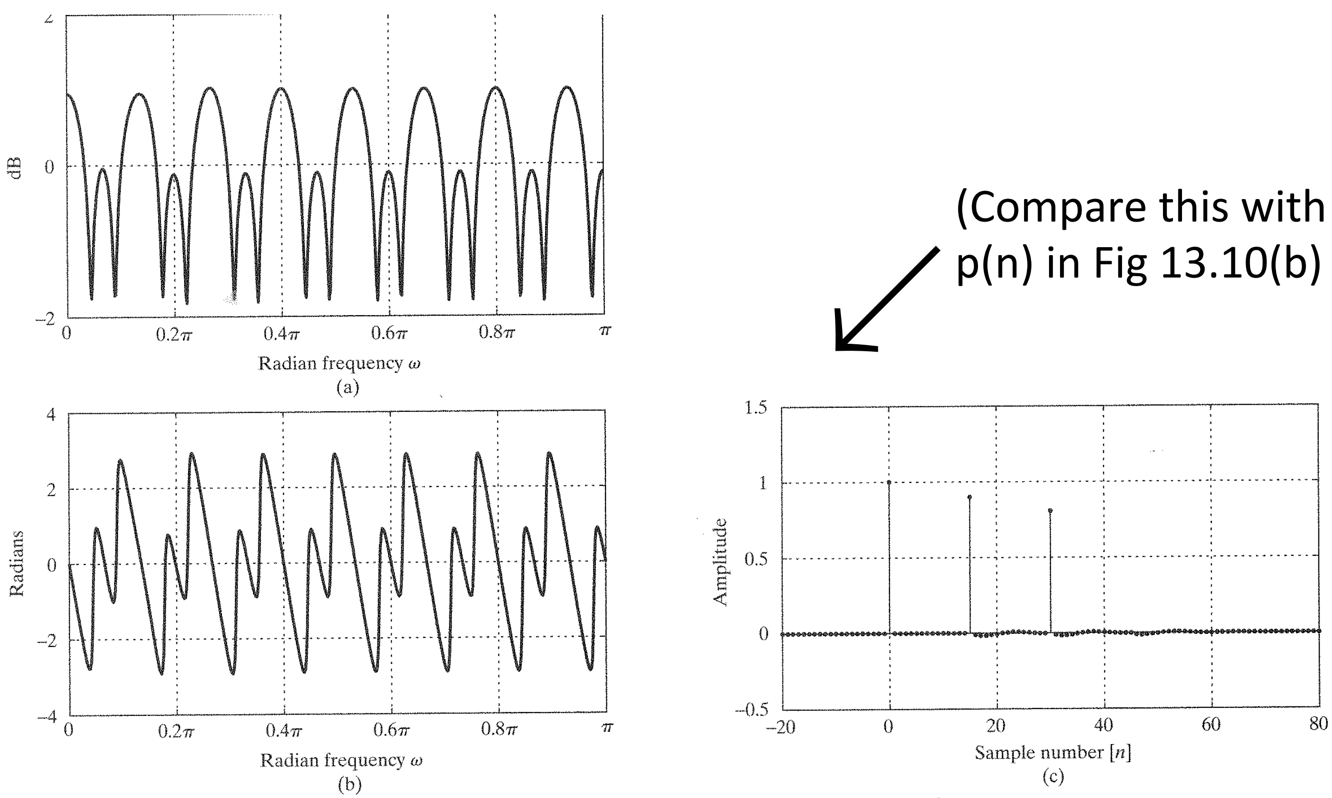


Figure 13.18 Illustration of highpass frequency-invariant linear filtering in the system of Figure 13.15. (a) Real part of the Fourier transform of the output of the highpass frequency-invariant system with $N_1 = 14$ and $N_2 = 512$ in Figure 13.16(b). (b) Imaginary part for conditions of part (a). (c) Output sequence $y[n]$ for the input of Figure 13.10.

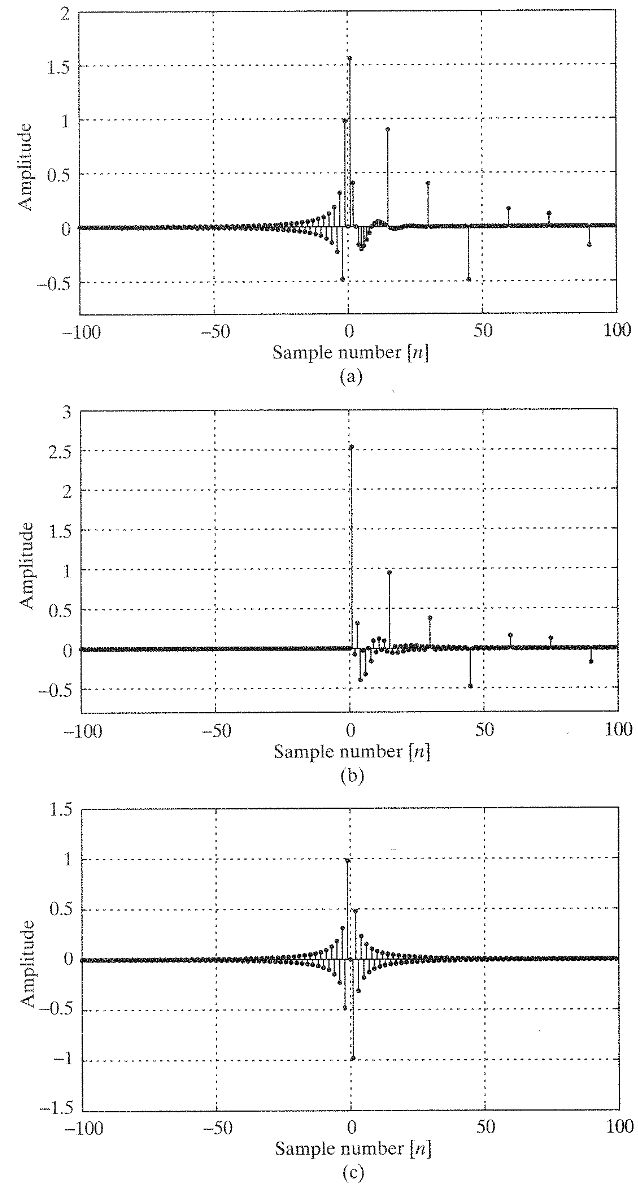


Figure 13.19 (a) Complex cepstrum of $x[n] = x_{min}n * x_{ap}[n]$. (b) Complex cepstrum of $x_{min}[n]$. (c) Complex cepstrum of $x_{ap}[n]$.

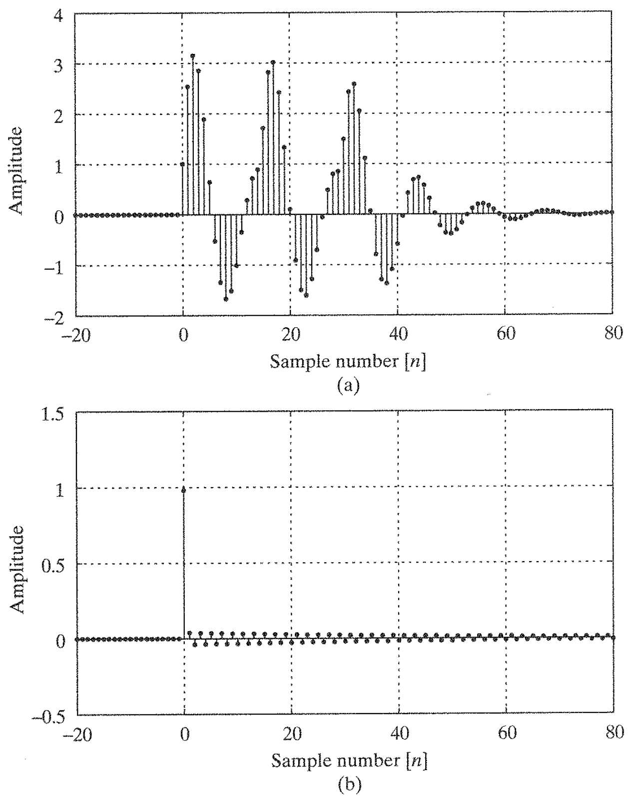


Figure 13.20 (a) Minimum-phase output. (b) Allpass output obtained as depicted in Figure 13.7.

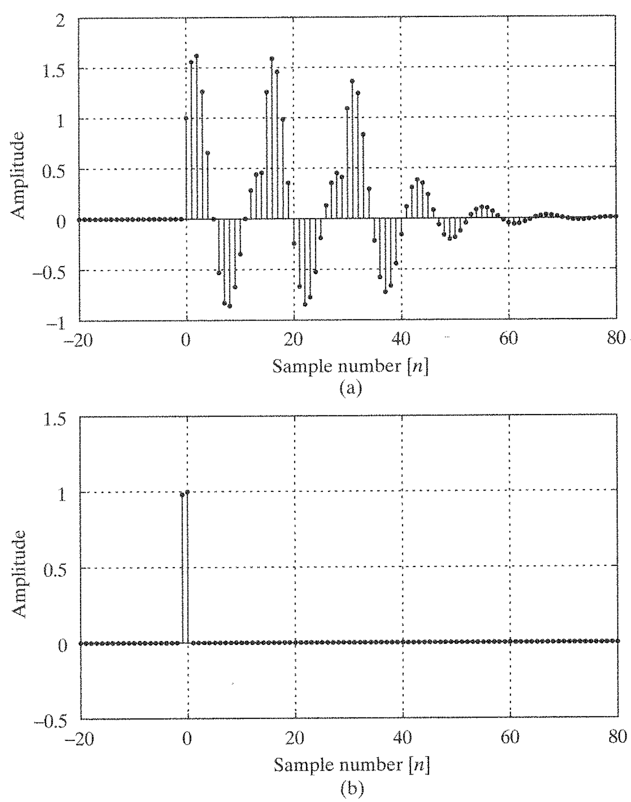


Figure 13.21 (a) Minimum-phase output. (b) Maximum-phase output obtained as depicted in Figure 13.8.