

Applications of modular forms to partitions and multipartitions

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Partitions

Multipartitions

Classical Results for $p(n)$

Some Modern Results (utilizing modular form theory)

Why Modular Forms?

Introduction to Modular Forms

Outline

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Goal

The goal of this talk is to highlight some applications of the theory of modular forms to problems in partition theory.

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Goal

The goal of this talk is to highlight some applications of the theory of modular forms to problems in partition theory.

First we introduce some partition theoretic objects of interest:

Partitions

A *partition* π of a positive integer n is a nonincreasing sequence of positive integers $\lambda_0, \dots, \lambda_m$ that sum to n . We write $n = |\pi|$, to denote the size of a partition, and call the λ_i the *parts* of the partition π .

In addition, we allow \emptyset to be considered as the unique partition of 0.

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In addition, we allow \emptyset to be considered as the unique partition of 0.

For example,

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

are all of the partitions of 5.

Graphical Representation

We can represent a partition π of n graphically by a Ferrers-Young diagram, which is a left-justified array of n dots, each row representing a part of π in nonincreasing order from top to bottom.

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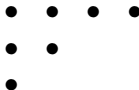
For example, the diagram for the partition $3 + 2 + 1 + 1$ of $n = 7$ is



The *conjugate* π' of a partition π of n is the partition of n obtained from π by reading the columns instead of the rows in the Ferrers-Young diagram.

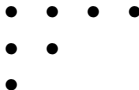
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Notice that the diagram of π' is the reflection of the diagram for π .

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For example,

$$(2 + 1, 3 + 3 + 1, \emptyset, 3 + 2 + 2)$$

is a 4-partition of 17.

Partition Functions

Of particular interest are functions that count partitions. We define for nonnegative n

$$p(n) := \#\{|\pi| = n\}$$

$$t(n) := \#\{|\pi| = n \mid \mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod{4}\}$$

$$p_k(n) := \#\left\{(\pi_1, \dots, \pi_k) \mid \sum_{i=1}^k |\pi_i| = n\right\}$$

where $\mathcal{O}(\pi)$ is the number of odd parts in π .

These functions are all interesting in their own right as combinatorial objects. In addition, both $p(n)$, and $p_k(n)$ have connections to physics and representation theory. The function $t(n)$ is a function defined by R. Stanley which arose from a study of sign-balanced labeled posets.

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These functions carry with them some fascinating arithmetic properties, which can be unleashed through the applications of the theory of modular forms.

One of the most fascinating characteristics of $p(n)$ are the following congruences of Ramanujan: For all $n \geq 0$,

$$p(5n + 4) \equiv 0 \pmod{5}$$

$$p(7n + 5) \equiv 0 \pmod{7}$$

$$p(11n + 6) \equiv 0 \pmod{11}$$

Also, Hardy and Ramanujan proved an asymptotic formula for the growth of $p(n)$ that Rademacher later perfected: For all $n \geq 0$,

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dx} \left(\frac{\sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}(x - 1/24)}\right)}{\sqrt{x - 1/24}} \right)_{x=n}$$

(where $A_k(n)$ is a messy sum of exponentials involving dedekind sums).

S. Ahlgren and M. Boylan

Theorem

If $p(qn + b) \equiv 0 \pmod{q}$ for some prime q , then $(q, b) \in \{(5, 4), (7, 5), (11, 6)\}$. I.e., there are no other Ramanujan congruences for $p(n)$ beyond the classical cases.

However, if we extend to considering congruences of the form $p(an + b) \equiv 0 \pmod{m}$, then there are other congruences but they are not as glamorous. For example, one of the simplest examples is the congruence

$$p(17303n + 237) \equiv 0 \pmod{13}.$$

K. Ono, K. Ono and S. Ahlgren

Theorem

For a prime $q \geq 5$, a positive integer m , and any $0 \leq b \leq q - 1$ satisfying a certain numerical requirement (a total of $(q + 1)/2$ of them), there are infinitely many non-nested arithmetic progressions $\{An + B\} \subseteq \{qn + b\}$, such that for every $n \geq 0$ we have

$$p(An + B) \equiv 0 \pmod{q^m}.$$

H. S.

Theorem

Let $q \geq 5$ be prime, and m a positive integer. There are infinitely many arithmetic progressions $\{An + B\}$ such that for all $n \geq 0$ we have

$$t(An + B) \equiv p(An + B) \equiv 0 \pmod{q^m}.$$

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$$\lim_{n \rightarrow \infty} \frac{t(n)}{p(n)} = \frac{1}{2}.$$

S. Treneer

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Let $q \geq 5$ be prime, and m a positive integer. There are infinitely many arithmetic progressions $\{An + B\}$ such that for all $n \geq 0$ we have

$$p_k(An + B) \equiv 0 \pmod{q^m}.$$

Why are modular forms so useful in partition theory?

Generating Functions

The main tool to begin the study of a partition function is to find a generating function. The generating functions for the partitions we are considering are:

$$F(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n},$$
$$M(q) := \sum_{n=0}^{\infty} p_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^k},$$

and the generating function for $t(n)$ is a bit more complicated, so we describe it in terms of $F(q)$:

$$\sum_{n=0}^{\infty} t(n)q^n = \frac{F(q)F(q^4)^5F(q^{32})^2}{F(q^2)^2F(q^{16})^5}.$$

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$$\sum_{n=0}^{\infty} t(n)q^n = \frac{F(q)F(q^4)^5F(q^{32})^2}{F(q^2)^2F(q^{16})^5}.$$

Furthermore, when viewed analytically, these functions converge absolutely for $|q| \leq 1$.

The reason that these functions are prime candidates for the application of modular forms, is due to the fact that $F(q)$ looks very similar to the complex function

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).$$

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In particular, defining $q := e^{2\pi iz}$, we see that

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = \frac{q^{\frac{1}{24}}}{F(q)}.$$

The function $\eta(z)$ is a type of modular form.

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To understand these properties, we consider the group $\text{SL}_2(\mathbb{Z})$ of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and determinant 1, and its action on \mathcal{H} via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}.$$

The group $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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A fundamental domain for the action is given by

$$F := \{-1/2 \leq z < 1/2 \mid |z| > 1\} \cup \{-1/2 \leq z \leq 0 \mid |z| = 1\}.$$

There are subgroups of $\Gamma(1)$ also of particular interest, called *congruence subgroups*. In particular

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$$

Notice that our use of the notation $\Gamma(1)$ for $\mathrm{SL}_2(\mathbb{Z})$ makes sense with this definition.

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We also consider congruence subgroups Γ acting on $\mathbb{Q} \cup \infty$ where

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c} = \lim_{z \rightarrow \infty} \frac{az + b}{cz + d},$$

Adding in the equivalence classes of $\mathbb{Q} \cup \infty$ to the fundamental domain F of a congruence subgroup Γ gives a Riemann surface.

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More specifically,

Definition of Modular form

A holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is called a *modular form of integer weight k* if

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for all $z \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and that f is “holomorphic at the cusps of Γ ” meaning for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$,

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \sum_{n \geq n_0} a(n) e^{2\pi i n z / N}.$$

If a modular form f vanishes at the cusps, we call f a *cuspidal form*, and if f is merely meromorphic at the cusps, we call f a *weakly holomorphic modular form*.

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The sets $M_k(\Gamma)$ (and $S_k(\Gamma)$) of modular forms (or cusp forms) of weight k is a complex vector space. When $\Gamma = \Gamma(1)$, it is particularly easy to describe. Notice that for this group, there is only one cusp, which we call ∞ .

Let $q := e^{2\pi iz}$. Define

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where B_k is the k th Bernoulli number and $\sigma_k(n) = \sum_{1 \leq d|n} d^k$.

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Then for $k \geq 4$ even, we have that $E_k \in M_k(\Gamma(1))$. Notice that k must be even since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is in $\Gamma(1)$.

A basis for the vector space $M_k(\Gamma(1))$ over \mathbb{C} is

$$\{E_4(z)^a E_6(z)^b \mid 4a + 6b = k, a, b \geq 0\}.$$

The simplest example of a cusp form is $\Delta(z) \in S_{12}(\Gamma(1))$.

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This function is particularly interesting to us since

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \eta(z)^{24},$$

and this is the same $\eta(z)$ that we saw arising in the generating function for partitions!

Unfortunately $\eta(z)$ is not quite as nice of a modular form as $\Delta(z)$. In particular, it does not have integer weight. There is a theory however of half-integer weight modular forms. Moreover, from work of Shimura there is an explicit correspondence between modular forms of integer and half-integer weights.

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If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is the Fourier expansion at infinity of a modular form in $M_k(\Gamma_0(N))$, then

$$f(z)|T_k(p) := \sum_{n=0}^{\infty} (a(pn) + p^{k-1}a(n/p))q^n$$

is a modular form in $M_k(\Gamma_0(N))$ as well. Likewise $T_k(p)$ takes cusp forms to cusp forms.

This operator can be extended to have meaning for half-integer weight modular forms as well, and in particular it commutes with the Shimura correspondence.

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Theory of Serre (arising from Galois representations) gives conditions for when cusp forms must vanish under the Hecke operator modulo m . In particular, there are conditions for which infinitely many modular forms vanish *simultaneously*... this is what allows for the proof that for $q \geq 5$ be prime, and m a positive integer, there are infinitely many arithmetic progressions $\{An + B\}$ such that for all $n \geq 0$ we have

$$t(An + B) \equiv p(An + B) \equiv 0 \pmod{q^m}.$$