

## *Languages that are Not Regular*

*There are severe limits to the recognition problems that an FA can handle. But this requires proof.*

## *A Nonregular Language: $0^n 1^n$*

There is no FA for

$$B = \{0^n 1^n : n \geq 0\} = \{\varepsilon, 01, 0011, 000111, \dots\}$$

Informal argument: Suppose input known to be 0's followed by 1's. The FA has to **count** the 0's: that is, at end of 0's it must be in state unique to number of 0's read. But FA has only fixed finite memory, while there can be arbitrarily many 0's. Impossible.

## *(In)Distinguishable Strings*

Two strings  $x$  and  $y$  are **indistinguishable with respect to language  $L$**  if for every string  $z$ , it holds that  $xz \in L$  if and only if  $yz \in L$ . Otherwise they are **distinguishable**.

Example: Say  $A$  is the language of binary strings with odd number of 1's. Then strings `0` and `10001` are indistinguishable with respect to  $A$ .

## *Distinguishable Strings and States*

**Theorem.** *If  $M$  is DFA accepting language  $L$ , and  $x$  and  $y$  are distinguishable strings with respect to  $L$ , then  $M$  must be in a different state after reading  $x$  than after reading  $y$ .*

Suppose strings  $x$  and  $y$  put  $M$  in same state. Then for any string  $z$  the strings  $xz$  and  $yz$  put  $M$  in the same state. So either  $xz$  and  $yz$  are both in  $L$  or both are out. Hence  $x$  and  $y$  are indistinguishable.

## *Sets of Distinguishable Strings*

**Corollary.** *If  $\mathcal{D}_L$  is set of **pairwise** distinguishable strings with respect to  $L$ , then any DFA for  $L$  has at least  $|\mathcal{D}_L|$  states. In particular, if  $\mathcal{D}_L$  is infinite then  $L$  is not regular.*

*Example:*  $0^n 1^n$

Recall  $B = \{0^n 1^n : n \geq 0\}$ .

The set  $\mathcal{D}_B = \{0^j : j \geq 0\}$  is pairwise distinguishable. Well, take any two strings in  $\mathcal{D}_B$ : say  $0^j$  and  $0^{j'}$  with  $j \neq j'$ . Appending  $1^j$  to the first produces a string in  $B$ , but appending  $1^{j'}$  to the second produces a string not in  $B$ . That is,  $0^j$  and  $0^{j'}$  are distinguishable.

Since  $\mathcal{D}_B$  is infinite,  $B$  is not regular.

## *Example: Lower Bound on Number of States*

Consider the language of all binary strings with an even number of both 0's and 1's. The set  $\{\epsilon, 0, 1, 01\}$  is pairwise distinguishable. So the product construction DFA we built has the fewest states possible.

(In general, there is a connection between smallest number of states of a DFA and the largest set of pairwise distinguishable strings.)

## Practice

Define  $E$  as the language of all binary strings with an **equal** number of 0's and 1's.

Show that  $E$  is not regular by finding an infinite set of pairwise distinguishable strings.

## *Solution to Practice*

Let  $D_E$  be the set of all strings containing only 0's. Then for  $i \neq j$  the strings  $0^i$  and  $0^j$  are distinguishable with respect to  $E$ , since  $0^i 1^i \in E$  but  $0^i 1^j \notin E$ .

The set  $D_E$  is infinite, and so  $E$  is nonregular.

## *The Pumping Lemma*

The infamous Pumping Lemma says that every regular language has a certain repetitiveness about it:

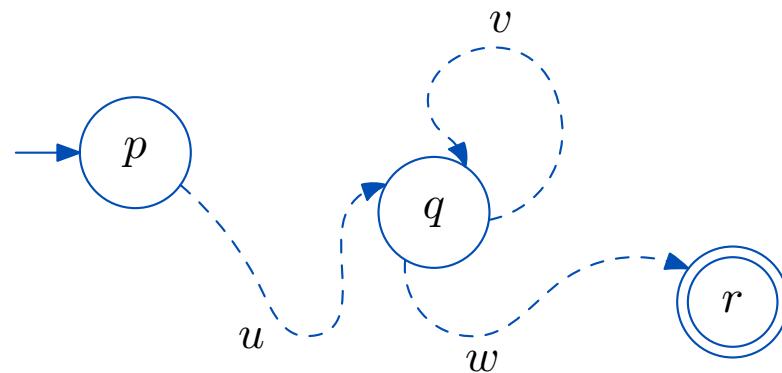
**Pumping Lemma.** *Let  $A$  be a regular language accepted by a DFA with  $k$  states. Then, for any string  $z$  in  $A$  with at least  $k$  symbols, one can find an early internal subsegment that can be pumped. That is,  $z$  can be split as  $uvw$  where:*

- $v$  is nonempty,
- $|uv| \leq k$ , and
- $uv^i w$  is in  $A$  for all  $i \geq 0$ .

## Proof of Pumping Lemma

Follow the sequence of states around DFA for  $A$  on input  $z$ . Let  $q$  be first of the  $k$  states to recur.

Then split the string  $z$  as follows:



It follows that the DFA is in same state  $r$  no matter whether it has read  $uw$ ,  $uvw$  or  $uv^2w$ . Hence,  $uv^i w$  is in  $A$  for all  $i \geq 0$ .

## *Using the Pumping Lemma*

The Pumping Lemma is used to show that a language is nonregular, by showing that the lemma is contradicted.

One needs to choose **one** string  $z$  that does not pump.

*Example:*  $0^n 1^n$

Consider again language  $B = \{ 0^n 1^n : n \geq 0 \}$ .

Suppose  $B$  were regular.

Then  $B$  would be accepted by DFA with  $k$  states.  
Consider the specific string  $z = 0^k 1^k$ . This is in  $B$ .

Split  $z = uvw$  according to Pumping Lemma.

Then since  $|uv| \leq k$ , it follows that  $v$  is composed entirely of  $0$ 's. But then  $uw$  is not in  $B$  (since has fewer  $0$ 's than  $1$ 's).

This contradicts the Pumping Lemma.

## Example: Palindromes

A **palindrome** is a word that reads the same backwards as it does forward (such as “level”). Let  $P$  be the set of all palindromes for alphabet  $\{a, b\}$ .  $P$  is nonregular.

Suppose  $P$  were regular. Then it would be accepted by DFA with say  $k$  states. Consider the string  $z = a^kba^k$ . Split  $z = uvw$  according to Pumping Lemma. Then since  $|uv| \leq k$ , it follows that  $v$  is just  $a$ 's. Thus  $uw$ , is not in  $P$ , a contradiction of the Pumping Lemma.

## Practice

Define  $E$  as the language of all binary strings with an **equal** number of 0's and 1's.

Show that  $E$  is not regular by showing that it contradicts the Pumping Lemma.

## *Practice Solution*

Suppose  $E$  were regular. Then it would be accepted by a DFA with say  $k$  states.

Consider specific  $z = 0^k 1^k$ . Split  $z = uvw$  according to the Pumping Lemma. Then because  $|uv| \leq k$ ,  $v$  is always just 0's. Thus  $uw$  is not in  $E$ .

This contradicts the Pumping Lemma.

## *Overview*

To show that a language is nonregular, you can show that there is an infinite set of pairwise distinguishable strings. Or you can use the Pumping Lemma and show that there is some string that cannot be pumped.