In-class Practice 16: Pumping Lemma

Let C be the language of all strings of $\mathtt{a}$'s, $\mathtt{b}$'s, $\mathtt{c}$'s and $\mathtt{d}$'s, such that the number of $\mathtt{a}$'s equals the number of $\mathtt{b}$'s, and the number of $\mathtt{c}$'s equals the number of $\mathtt{d}$'s.

Prove that C is not context-free.

Suppose that C is context-free. Then let k be the constant of the Pumping Lemma. Choose the string

$$z = \mathtt{a}^k\mathtt{c}^k\mathtt{b}^k\mathtt{d}^k$$

The string z is in the language C, and is sufficiently long for the Pumping Lemma to apply.

Consider the split of the string z into $uvwxy$ (that the Pumping Lemma says exists). Because vwx combined has length at most k, the string vx can contain symbols from at most two of the blocks (and these blocks must be next to each other). So if vx contains an $\mathtt{a}$, then it cannot contain a $\mathtt{b}$, and if vx contains a $\mathtt{c}$, then it cannot contain a $\mathtt{d}$, and vice versa.

In every case this means that $z^{(0)} = uwy$ either does not have equal $\mathtt{a}$'s and $\mathtt{b}$'s, or does not have equal $\mathtt{c}$'s and $\mathtt{d}$'s (or both). That is, $z^{(0)}$ is never in the language C, a contradiction.