Parametric Vector Form of the Solution

ALGOR *The solution to a general linear system can be written in* ***parametric vector form*** *as: one vector plus an arbitrary linear combination of vectors satisfying the corresponding homogeneous system.*

Defintion

Defn. *A collection of vectors is **linearly independent** if the only linear combination of them that equals* 0 *is the trivial combination (all weights zero). Otherwise it is said to be **linearly dependent**.*

Note that the collection is linearly dependent if some vector in it can be written as a linear combination of the other vectors.

Key Examples

$\succ$ Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent if and only if one is a multiple of the other. If they are linearly independent, then they span a plane through the origin. Further, inserting $\boldsymbol{w}$ into the collection produces a linearly independent set if and only if $\boldsymbol{w}$ is not in $Span\{\boldsymbol{u}, \boldsymbol{v}\}$.

$\succ$ A set containing the zero vector is automatically linearly dependent.

When Homogenous System has Unique Solution?

Fact. *The columns of matrix A are linearly independent*
$\iff$ $Ax = \mathbf{0}$ *has only the trivial solution*
$\iff$ *there is no free variable.*

Matrix Entries

Defn. *For matrix A, notation a_{ij} means the entry in row i and column j of A.*

Matrix Addition and Scalar Multiplication

Defn. *Matrix* ***addition*** *requires the two matrices have the same dimensions. The sum is defined by adding corresponding entries. Similarly,* ***scalar multiplication*** *is defined entry-wise.*

For example,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+a_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+a_{22} & a_{23}+b_{23} \end{bmatrix}$$

Matrix Transpose

Defn. *The **transpose** of a matrix A, denoted A^T, exchanges rows and columns. That is, $(A^T)_{ij} = A_{ji}$.*

For example: here is a matrix and its transpose

$$\begin{bmatrix} 3 & 4 & 7 \\ -2 & 5 & -3 \end{bmatrix} \qquad \begin{bmatrix} 3 & -2 \\ 4 & 5 \\ 7 & -3 \end{bmatrix}$$

Square Matrices

Defn. *A* ***square*** *matrix has equal number of rows and columns.*

Diagonal Matrices

Defn. *The* ***diagonal*** *of a square matrix runs from top-left to bottom-right. A* ***diagonal matrix*** *is a square matrix that has zeros off the diagonal (and might or might not have zeroes on the diagonal).*

For example

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\pi \end{bmatrix}$$

Matrix Multiplication via Matrix-Vector Mult

Defn. *If matrix* A *is* $m \times n$ *and matrix* B *is* $r \times s$, *then for the* ***product*** AB *to be valid it must be that* $n = r$. *If valid, the product* AB *has size* $m \times s$. *The columns of the product are the results of multiplying the first matrix by the columns of the second.*

That is,

$$AB = \left[A\boldsymbol{b}_1 \; A\boldsymbol{b}_2 \; \cdots \; A\boldsymbol{b}_s \right]$$

where $\boldsymbol{b}_j$ is the j^{th} column of B.

Example of Matrix Multiplication

The product of a 2×3 and 3×4 matrix is a 2×4 matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 & 5 \\ -2 & 0 & 3 & -4 \\ 1 & -2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 3 & -2 \\ -8 & 4 & 5 & -10 \end{bmatrix}$$

An example detail: the 3rd column of the result is given by $-\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Formula for Entry in Product

Note

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

That is, to calculate entry in row i and column j of the product, look at row i of the first matrix and column j of the second matrix; then multiply corresponding entries and add.

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} \bullet & \bullet & -1 & \bullet \\ \bullet & \bullet & 3 & \bullet \\ \bullet & \bullet & 2 & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & 5 & \bullet \end{bmatrix}$$

$$(0 \times -1) + (3 \times 3) + (-2 \times 2)$$

Matrix Multiplication is Associative

Fact. *Brackets don't matter.*

For example $(AB)C = A(BC)$ (and the one product is valid whenever the other one is).

Matrix multiplication is not Commutative

Fact. *Order matters.*

There is no guarantee that (and it is unlikely that) $AB = BA$.

Indeed, the one product might be valid when the other one is not.

The Identity Matrix

Defn. *The* ***identity matrix*** I_n *is the* $n \times n$ *diagonal matrix with* 1*'s on the diagonal. (We sometimes write just* I*.) Its columns are the vectors* $\mathbf{e}_i$*: these have* 0*'s in every position except for a* 1 *in the* i^{th} *position.*

If A is a square matrix, then $IA = AI = A$, where I is the identity matrix of the same size.

Matrix Powers

We use A^p to mean the product of p copies of A.
(This needs A to be square.)

Transpose and Products

Fact. $(AB)^T = B^T A^T$

Note that the order is swapped!

Elementary Row Operations Revisited

Fact. *Each elementary row operation is equivalent to multiplying on the left by a matrix called an* ***elementary matrix****.*

Matrix Transform

Defn. *If the matrix A is $m \times n$, then the* ***matrix transform*** $x \mapsto Ax$ *has domain $\mathbb{R}^n$, codomain $\mathbb{R}^m$, and range some subset of $\mathbb{R}^m$.*

For example, if A is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then the transform maps $(5, 3)$ to $(3, 5)$.

The Inverse of a Matrix

Defn. *The* ***inverse*** *of a square matrix A, denoted A^{-1}, is the matrix such that $AA^{-1} = A^{-1}A = I$.*

Defn. *The inverse is not guaranteed to exist. If it exists, then A is* ***invertible****; otherwise A is* ***not invertible*** *or* ***singular****.*

Matrix Equation with Invertible Matrix

Fact. *If matrix A is invertible, then $A\boldsymbol{x} = \boldsymbol{b}$ has unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.*

Inverse of a 2×2 *Matrix*

The inverse of a 2×2 matrix has formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The formula also captures when the inverse exists: the matrix is invertible if and only if $ad - bc \neq 0$.

Obtaining Matrix Inverses by Reduction

One way to find the inverse is to solve the collection of n vector equations $A\boldsymbol{x} = \boldsymbol{e}_1, \ldots,$ $A\boldsymbol{x} = \boldsymbol{e}_n$ (where the $\boldsymbol{e}_j$ are the columns of I_n as before). Equivalently:

ALGOR *To find inverse of matrix A, augment with the identity matrix I_n, then bring to reduced row echelon form.*

Formulas

Fact. *If A and B are square matrices of the same size:*

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1} A^{-1}$ *(Note the reversal!)*

(c) $(A^T)^{-1} = (A^{-1})^T$.

Characterization of Invertible Matrices

The big theorem.

Fact. *For an $n \times n$ matrix A, the following are* ***equivalent****:*

- A *is invertible*
- A *has* n *pivots*
- A *is row equivalent to* I_n
- $A\boldsymbol{x} = \mathbf{0}$ *has a unique solution*
- *the columns of* A *are linearly independent*
- *the columns of* A *span* $\mathbb{R}^n$
- *the range of transform* $x \mapsto Ax$ *is all of* $\mathbb{R}^n$

Block-Diagonal Matrices

Defn. *A* ***block-diagonal*** *matrix is one where all blocks off the diagonal are zero.*

Fact. *A block-diagonal matrix is invertible if and only if all the diagonal blocks are invertible. Moreover, its inverse is the block-diagonal matrix with the inverses of the diagonal blocks.*

Example Inverse of Block-Diagonal Matrix

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & -5 \\ 0 & -5 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 9/2 & 5/2 \\ 0 & 5/2 & 3/2 \end{bmatrix}$$

Triangular Matrices

Defn. *A **lower triangular** matrix is one whose entries above the main diagonal are zero. An **upper triangular** matrix is defined similarly.*

For example, a diagonal matrix is both lower and upper triangular.

Fact. *A square triangular matrix is invertible if and only if every entry on the diagonal is nonzero.*