Warmup 2: Functions, Relations, Proofs and Factors

1. Let $X$ be a set with 1000 elements and $Y$ a set with 1001 elements.

   (a) How many functions from $X$ to $Y$?  
       \[ 1001^{1000} \]

   (b) How many functions from $Y$ to $X$?  
       \[ 1000^{1001} \]

   (c) How many one-to-one functions from $X$ to $Y$?  
       \[ 1001! \]

   (d) How many one-to-one functions from $Y$ to $X$?  
       \[ 0 \]

2. Consider the universe of all 6-letter English words. In each case, state whether it is an equivalence relation or not, and if not, state one of the three properties it fails:

   (a) Define two words to be related if they end with the same letter
       
       Equivalence relation

   (b) Define two words to be related if they both contain at least two vowels
       
       Not reflexive: e.g. “tracks” is not related to itself

   (c) Define two words to be related if the first one occurs before the second one in dictionary order.
       
       Neither reflexive nor symmetric

3. Give a proof by contradiction that $\sqrt[3]{2}$ is irrational.

   Suppose that $\sqrt[3]{2}$ is rational. Then it equals $\frac{p}{q}$ where $p$ and $q$ are positive integers and have no common factor. Cubing both sides we get that $2 = \frac{p^3}{q^3}$ and thus $2q^3 = p^3$. So $p^3$ is even; so $p$ is even. Say $p = 2r$ for positive integer $r$. Then plug in we get $2q^3 = 8r^3$ and so $4r^3 = q^3$. So $q^3$ is even; so $q$ is even. But this is a contradiction, since we know that $p$ and $q$ have no common factor and yet we’ve just shown that 2 is a common factor.
4. Recall that $2023 = 7 \times 17 \times 17$.

(a) How many distinct factors does 2023 have?

(b) What is the largest factor of $2023^{2023}$ apart from itself?

(c) What is the smallest factor of $2023^{2023}$ apart from 1?

\[2 \times 3 = 6\]

\[\frac{2023^{2023}}{7}\]

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