1. Prove by induction that for every nonnegative integer \( n \), the sum \( 5^{2n+1} + 9^{2n+1} \) is divisible by 7.

Base case \( n = 0 \). Expression = 5 + 9 = 14, which is a multiple of 7. Assume true for \( n = k \), test for \( n = k + 1 \). \( 5^{2(k+1)+1} + 9^{2(k+1)+1} = 25 \times 5^{2k+1} + 81 \times 9^{2k+1} = 21 \times 5^{2k+1} + 77 \times 9^{2k+1} + 4(5^{2k+1} + 9^{2k+1}) \), which is a multiple of 7 because every term is.

2. What is the gcd of the Fibonacci numbers \( f(2023) \) and \( f(2022) \)? Justify your answer.

   1. By Euclid \( \gcd(f(2023), f(2022)) = \gcd(f(2022), f(2021)) = \ldots = \gcd(f(1), f(0)) = 1 \)

3. (a) Let \( a(n) \) be the number of strings of 0’s and 1’s of length \( n \) that do not contain three 0’s in a row. Give a recurrence for \( a(n) \).

   (b) Let \( b(n) \) be the number of strings of 0’s, 1’s, and 2’s of length \( n \) that do not contain two 1’s in a row. Give a recurrence for \( b(n) \).

   (c) Let \( c(n) \) be the number of strings of 0’s, 1’s, 2’s and 3’s that do not contain a 2 immediately followed by a 3. Give a recurrence for \( c(n) \).

   \[
   a(n) = a(n-1) + a(n-2) + a(n-3).
   
   b(n) = 2b(n-1) + 2b(n-2).
   
   c(n) = 3c(n-1) + 2c(n-2) + 2c(n-3) + \ldots + 2c(2) + 3c(1)
   \]

4. (a) A recursive definition of a rooted tree is that (i) a single root is a rooted tree, and (ii) adding \underline{a child node to an existing node} yields a rooted tree.

   (b) If a graph has the property that between every two vertices there is a \underline{path}, then it is a tree.

   (c) One definition of a tree is that it is a graph that is \underline{unique} and contains \underline{connected \ldots no cycle}.

   (d) A tree with 2023 vertices has \underline{2022} edges.

5. Call a rooted tree \textit{gorgeous} if every vertex has an even number of children. Draw all gorgeous rooted trees with 7 vertices, assuming vertices are indistinguishable and the order of children doesn’t matter.