1. It is known that the number of spanning trees of the complete bipartite graph $K_{r,s}$ is $r^{s-1}s^{r-1}$.

(a) Apply the matrix-tree theorem to verify this for $K_{3,3}$. (Software permitted.)
(b) Prove from first principles that the formula is true for $r = 2$ and general $s$.
(c) Propose a Prüfer code for complete bipartite graphs and give an example. (Do not have to prove all details.)

2. Show that the vertex set of a graph of maximum degree $\Delta$ can be partitioned into $\lfloor \Delta/2 \rfloor + 1$ forests.

3. Suppose that $G$ is 100-regular and $\kappa(G) = 1$. What is the minimum and maximum value of $\kappa'(G)$? Justify your answer.

4. Prove that a 3-regular graph has a perfect matching if and only if it decomposes into copies of $P_4$.

5. (a) Show that a tree has a perfect matching if and only if its independence number is half the order.
(b) Call a graph happy if every spanning tree has a perfect matching. Give an example of a nonbipartite happy graph of order 2024.
(c) Show that no 3-connected graph is happy.

6. An induced matching is a collection of edges where the graph induced by the vertices incident with the edges is 1-regular. Calculate the maximum size of an induced matching in:
(a) $K_n$
(b) $P_{2024}$
(c) The Petersen graph

Due: Thursday February 15