D A Little Linear Algebra

Linear algebra shows up in graph theory in many ways. For example, the eigenvalues of the graph provide information about the graph. Here we do two small examples.

D1 Counting Walks

Recall that in a walk one can repeat vertices and edges. Number the vertices from 1 up to \( n \), and let \( w_k(i,j) \) be the number of walks from \( i \) to \( j \) of length \( k \). For example, the number of walks from \( i \) to \( i \) of length 2 is the degree of \( i \).

**The Goodwill–Hunting Theorem.** The value \( w_k(i,j) \) equals the \((i,j)\)-entry of \( A^k \), where \( A \) is the adjacency matrix.

**Proof.** We can prove this by induction on \( k \). The base case is \( k = 1 \), which is true, since the adjacency matrix \( A^1 \) by definition gives the number of length-1 walks. In general, assume we know it is true for \( k \) and test for \( k + 1 \). Each walk from \( i \) to \( j \) of length \( k + 1 \) can be split into a walk of length \( k \) from \( i \) to some vertex \( \ell \) followed by the edge \( \ell \) to \( j \). So the total number is

\[
    w_k(i,j) = \sum_{\ell=1}^{n} w_k(i,\ell)w_1(\ell,j) = \sum_{\ell=1}^{n} A_{i,\ell}^k A_{\ell,j} = (A^k A)_{i,j} = A_{i,j}^{k+1},
\]

as required. \( \square \)

D2 Cages

The next result is about the existence of particular graphs. Consider an \( r \)-regular graph with girth 5 (no triangle nor 4-cycle). Then a vertex, its neighbors, and their neighbors must all be distinct. This means there must be at least \( 1 + r + r(r - 1) = r^2 + 1 \) vertices. The question is, when is there such a graph with exactly \( r^2 + 1 \) vertices? Such a graph is called a **cage**

For example, when \( r = 2 \) one has \( C_5 \), and when \( r = 3 \) one has the Petersen graph. Using matrix algebra, we can almost solve this question.
**Theorem.** If there is an \( r \)-regular graph on \( r^2 + 1 \) vertices with girth 5, then \( r \) must be one of 2, 3, 7, or 57.

**Proof.** Let \( G \) be such a graph. Then the conditions mean that the diameter of \( G \) is 2. In fact, every two vertices are either adjacent or have a unique common neighbor. That is, if \( i \) and \( j \) are adjacent there is no walk of length 2 between them, and if \( i \) and \( j \) are not adjacent there is exactly one walk of length between them.

So by the above theorem this means that the matrix \( A^2 + A \) has \( r \)'s on the diagonal and 1's everywhere else. As a matrix equation we have:

\[
A^2 + A - (r - 1)I = J,
\]

where \( J \) is the all-1 matrix.

Now let \( \lambda \) be an eigenvalue of \( A \) with eigenvector \( v \). The above equation implies that \( v \) is an eigenvector of \( J \) with eigenvalue \( \lambda^2 + \lambda - (r - 1) \). A common exercise in a linear algebra class is to determine the eigenspaces of \( J \). It can be checked that the matrix \( J \) has eigenvalue \( n \) with multiplicity 1 and the all-1-vector as eigenvector; and eigenvalue 0 with multiplicity \( n - 1 \).

The all-1 vector is an eigenvector of \( A \) with eigenvalue \( r \). The above discussion means all other eigenvalues of \( A \) satisfy \( \lambda^2 + \lambda - (r - 1) = 0 \). This equation has roots

\[
\lambda^+ = \frac{-1 + \sqrt{1 + 4(r - 1)}}{2} \quad \text{and} \quad \lambda^- = \frac{-1 - \sqrt{1 + 4(r - 1)}}{2}.
\]

Now, suppose \( \lambda^+ \) has multiplicity \( b \) as eigenvalue of \( A \). It follows that \( \lambda^- \) has multiplicity \( n - 1 - b \). Recall from linear algebra that the sum of the eigenvalues with multiplicities is the trace, which for \( A \) is clearly 0. So we have that

\[
b\lambda^+ + (r^2 - b)\lambda^- + r = 0.
\]

Thus we have an equation for \( b \), but \( b \) must be an integer. Some algebra later we get that \( r \) must be one of the values given in the statement of the theorem.

There is a graph known for the case \( r = 7 \). But the case \( r = 57 \) remains unresolved.