F Vertex Colorings

F1 Colorings

A coloring of a graph is a labeling of the vertices with labels/colors. A coloring is *proper* if every pair of adjacent vertices receive different colors; equivalently, the set of vertices of any particular color form an independent set. A graph is *k-colorable* if there is a proper coloring using a universe of size *k*. The *chromatic number* $\chi(G)$ is the least $k$ such that the graph is $k$-colorable. For example, a graph being bipartite is equivalent to being 2-colorable; the Petersen graph is 3-colorable.

There is an immediate upper bound:

**Greedy.** For a graph of maximum degree $\Delta$, the chromatic number is at most $\Delta + 1$.

**Proof.** Armed with a palette of $\Delta + 1$ colors, simply greedily color the graph. When come to color a vertex, one needs to give it a color that is different from each of the colors already present in its neighborhood. So at most $\Delta$ colors cause a problem, and there is at least one color available. $\square$

For a lower bound:

**Lower bound.** (a) The chromatic number is at least the clique number.
(b) The chromatic number is at least $n/\alpha(G)$, where $n$ is the order and $\alpha(G)$ the independence number.

**Proof.** (a) Every vertex of the clique needs a different color.
(b) Every color can be used at most $\alpha(G)$ times. $\square$

A harder result is that the upper bound can be improved for most graphs; proof is given in next section.

**Brooks’ Theorem.** Except for cliques and odd cycles, $\chi \leq \Delta$.

The *cartesian product* of two graphs $G$ and $H$, written $G \square H$, is the graph with vertex set all pairs $(u, v)$ of $V(G) \times V(H)$ where two pairs are adjacent if
they are the same in one coordinate and adjacent in the other. For example: hypercube $Q_k = Q_{k-1} \square K_2$. The cartesian product has a spanning subgraph consisting of copies of $G$ and one consisting of copies of $H$. These copies are called **fibers**.

**Cartesian product.** $\chi(G \square H) = \max(\chi(G), \chi(H))$.

**Proof.** Since the cartesian product contains both initial graphs, its chromatic number is at least the maximum of the two. On the other hand, say the maximum of the two is $k$ and consider an optimal coloring $g$ of $G$ and $h$ of $H$; then assign $(u, v)$ the color $g(u) + h(v) \mod k$. One can then check that adjacent vertices in the product get different colors. □

## F2 Proof of Brooks

We first prove the result for a 2-connected graph. The proof idea: do a greedy coloring but ensure when get to last vertex it has two neighbors of the same color. We need a “nice” spanning tree:

**Lemma.** If $G$ is a 2-connected graph that is not complete nor a cycle, then there exists a pair of vertices $x$ and $y$ at distance 2 such that $G - \{x, y\}$ is connected.

**Proof.** We know $G$ contains a cycle. If that cycle is not the whole of $G$, then there exists a vertex outside the cycle, and by the connectedness, that vertex is joined to the cycle by two internally disjoint paths, say meeting at $u$ and $v$. Then there are three internally disjoint $u-v$ paths. Similarly, if the cycle is all of $G$, then since the graph is not a cycle there is a chord $uv$.

Now, out of all such structures (meaning two vertices joined by three internally disjoint paths), choose one that contains the most vertices. At least two of the paths have length more than 1; let $x$ and $y$ be the neighbors of $u$ on these paths. We claim that $G - \{x, y\}$ is connected. Note that what remains of the structure is connected (since can reach $v$). If some vertex $z$ is disconnected from the structure, it must have disjoint paths to $x$ and $y$. Then we can add these
paths to the structure: and now we have three internally disjoint $x$–$v$ paths, a contradiction. See picture.

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**Lemma.** If $G$ is a 2-connected graph that is not complete nor an odd cycle then \( \chi \leq \Delta \).  

**Proof.** If $G$ is an even cycle, then $\chi = \Delta = 2$. Otherwise, by the above lemma there is a pair of nonadjacent vertices $x$ and $y$ with a common neighbor $w$ such that $G - \{x, y\}$ is connected.

Order the vertices of $G - \{x, y\}$ in decreasing distance from $w$ as measured in $G - \{x, y\}$. Say the vertices are $v_1, v_2, \ldots, v_{n-2}$ where $v_{n-2} = w$. Note that every vertex has a neighbor that is closer to $w$. So for each $i < n - 2$ there exists a $j_i > i$ such that $v_i$ and $v_{j_i}$ are adjacent.

Now we color $G$ using the colors \( \{1, 2, \ldots, \Delta\} \). Start by coloring $x$ and $y$ with color 1. (They are not adjacent.) Then color the $v_i$ in increasing order. For each $v_i$ except the final one, when we color it at least one of its neighbors (namely $v_{j_i}$) is uncolored; thus there are at most $\Delta - 1$ colors among its neighbors, and so there is at least one color that we can assign to $v_i$. For $w$, all of its neighbors are colored before it; but two of them (namely $x$ and $y$) get the same color and so there is at least one color that is not present among $w$’s neighbors that we can use for $w$. \( \square \)

Finally we prove the theorem for general graphs. The proof is by induction.

If $G$ is 2-connected then we are done. So suppose the removal of vertex $v$ disconnects $G$ into two connected components. (Or more than two, it doesn’t really matter.) Let $G_1$ and $G_2$ be the two connected graphs each consisting of one of these components together with the vertex $v$. Note that the chromatic number of $G$ equals $\max\{\chi(G_1), \chi(G_2)\}$, since one can take an optimal coloring of each $G_i$ and rename colors if needed so that $v$ has the same color in both.
We claim that $\chi(G_i) \leq \Delta(G)$. If $G_1$ is neither an odd cycle nor complete, then this holds by the induction hypothesis. If $G_1$ is an odd cycle or complete, then $\chi(G_1) = \Delta(G_1) + 1$. But $\Delta(G_1) < \Delta(G)$, since the vertex $v$ has maximum degree in $G_1$ and its degree in $G$ is bigger than its degree in $G_1$.

**F3 Extremal Ideas**

*Mycielski construction:* Can obtain triangle-free graphs with arbitrarily large chromatic number. The idea is to start with $G_2 = K_2$ and construct $G_{i+1}$ from $G_i$ by cloning all the vertices but having no edges between the new vertices, and adding a final vertex. The graph $G_4$ is known as the Grötzsch graph.

A graph is *$k$-critical* if the chromatic number is $k$ but decreases on the removal of any edge. So the 3-critical graphs are the odd cycles. It is easy to show that the minimum degree of a $k$-critical graph is at least $k - 1$. The above shows that having chromatic number large does not require a large clique. But we can say something:

**Dirac.** Every graph with chromatic number at least 4 contains a subdivision of $K_4$.

We saw Manvel’s theorem earlier. Here is an extension that can be proved.

**Turán.** The maximum number of edges in a graph that has no $K_{k+1}$ is achieved by taking a complete $k$-partite graph with as equal parts as possible.

Where a complete multipartite graph is the simple graph where the vertex set can be partitioned into subsets such that each subset is an independent set and any two vertices in different subsets are adjacent.

**F4 Chordal Graphs and Perfect Graphs**

There are several ways to produce a graph by taking a collection of objects and making each object a vertex and defining adjacency if the two objects are related
some how. For example, an *interval graph* is one that can be obtained from a collection of intervals that are adjacent if they overlap.

**Interval Coloring.** *The chromatic number of the interval graph is its clique number.*

**Proof.** Order the intervals by their left-most endpoint and run the greedy algorithm coloring the intervals. If one needs color $k$ for a particular interval, then it overlaps an interval with each of color 1 through $k - 1$; but it and these intervals must form a clique. □

An interval graph is a special case of chordal graphs. A *chordal graph* is defined to be a simple graph that does not have a chordless cycle (meaning an induced cycle of length at least 4). It can be shown that:

**Elimination Ordering.** *A graph is chordal if and only if there is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that if we successively delete vertices in order, at the moment of deletion the neighborhood of $v_i$ is complete.*

As a consequence we get that chordal graphs have chromatic number equal to clique number. (Do the greedy algorithm but process the vertices $v_n, \ldots, v_1$.)

It can also be shown that:

**Characterization.** *A graph is chordal if and only if it is the intersection graph of subtrees of a tree.*

If a graph and all its induced subgraphs have chromatic number equal to clique number, then the graph is called *perfect*. A chordal graph is a special case of a perfect graph.