Hamiton Cycles and Beyond

We define a Hamilton cycle as one that contains all the vertices; similarly a Hamilton path is one that contains all the vertices.

**Dirac’s theorem.** If $G$ is a graph with order $n \geq 3$ and minimum degree $\delta \geq n/2$, then $G$ contains a Hamilton cycle.

**Proof.** Suppose there is a counterexample, meaning a graph with the minimum degree condition that does not have a hamilton cycle. Then consider the counterexample $G$ with the maximum number of edges. Clearly $G$ is not complete; say it is missing edge $uv$. So consider the graph $G + uv$. By our discussion, this has a Hamilton cycle, and that Hamilton cycle uses edge $uv$. So in $G$ there is a Hamilton path $P$ from $u$ to $v$.

Say $P$ is $u = x_1, \ldots, v = x_n$. Suppose there is $i$ with $2 \leq i \leq n - 2$ where the pair $Q_i = \{ux_{i+1}, vx_i\}$ of edges both exist; then we get a cycle using all vertices of $G$.

\[
\begin{array}{c}
 u \quad x_i \quad x_{i+1} \quad v \\
| \quad | \quad | \\
| \quad | \quad | \\
| \quad | \quad | \\
\end{array}
\]

The union of the $Q_i$ accounts for all the edges incident with $u$ or $v$, except for $ux_2$ and $vx_{n-1}$. Thus the total number of edges incident with them is at most $n - 1$. But that contradicts the minimum degree condition. $\blacksquare$

This theorem is best possible. Here are two graphs with minimum degree $(n - 1)/2$ that do not have a Hamilton cycle: $K_{m,m+1}$; or two copies of $K_m$ with one vertex of each identified. There are many extensions. The first was by Ore: if the degree-sum $d(u) + d(v) \geq n$ for all nonadjacent vertices $u$ and $v$, then the graph has a Hamilton cycle.

**Chvátal–Erdős.** If $G$ is a graph with order at least 3 and the connectivity is at least the independence number, then the graph has a Hamilton cycle.

**Proof.** If the independence number is 1, then the graph is complete. So assume the independence number $k \geq 2$. In particular this means the connectivity is at least 2. Consider the longest cycle $C$. Suppose there exists a vertex $w$ outside the cycle. By Menger’s Theorem, there are internally disjoint paths $P_1, \ldots, P_k$
from $w$ to $C$. If any two paths meet $C$ at consecutive vertices of $C$, then one can insert $w$ to get a longer cycle. So assume not.

For each path $P_ℓ$, let $v_ℓ$ be the vertex of $C$ after the end of $P_ℓ$ going clockwise. We claim that: the set $\{v_ℓ\}$ is an independent set. This follows because a chord $v_ℓv_ℓ'$ would enable a longer cycle using the chord, $P_ℓ$, and $P_ℓ'$.

But by the condition of theorem, adding $w$ to $\{v_ℓ\}$ yields a set that is not independent. This means that $w$ is adjacent to some $v_ℓ$, and one can thus insert $w$ using $P_ℓ$ and this edge. This contradicts the claim that $C$ is a longest cycle. □

For planar graphs, Tutte showed that 4-connected implies the existence of a Hamilton cycle. In general, a necessary condition for a Hamilton cycle is that the removal of any vertex set $S$ leaves at most $|S|$ components. The **toughness** of a graph is defined to be the minimum of the ratio $|S|/k(G - S)$ over all cut-sets $S$, where $k(G - S)$ is the number of components of $G - S$.

**Open Question.** Does a sufficiently large toughness guarantee a Hamilton cycle?

A graph is **pancyclic** if it contains cycles of each length from 3 up to its order. The minimum-degree threshold to guarantee a graph is pancyclic is only slightly larger than Dirac’s bound:

**Minimum Degree.** If $G$ is a graph with order $n ≥ 3$ and minimum degree $δ ≥ (n + 1)/2$, then $G$ is pancyclic.

However, if the graph is known to have a Hamilton cycle, then the threshold can be reduced:

**BFG.** If $G$ is a graph that contains a Hamilton cycle and a triangle and minimum degree $δ ≥ (n + 2)/3$, then $G$ is pancyclic.