Hamilton Cycles and Beyond

We define a Hamilton cycle as one that contains all the vertices; similarly a Hamilton path is one that contains all the vertices.

**Dirac’s theorem.** If $G$ is a graph with order $n \geq 3$ and minimum degree $\delta \geq n/2$, then $G$ contains a Hamilton cycle.

**Proof.** Consider the longest path $P$; say it goes from $u$ to $v$. Both $u$ and $v$ must have all their neighbors on $P$. By counting, either $u$ and $v$ are adjacent, or there are consecutive vertices $x, y$ on $P$ with edges $xv$ and $yu$. This yields a cycle $C$ using all vertices of $P$.

Suppose that $C$ is not a Hamilton cycle. Since all neighbors of $u$ are on $P$, the cycle $C$ must have at least $n/2 + 1$ vertices. Any vertex $w$ not on $C$ has $n/2$ neighbors, and so must have an edge joining it to $C$. But then one can create a longer path containing $w$ and all of $C$, a contradiction. \(\square\)

This theorem is best possible. Here are two graphs with minimum degree $(n - 1)/2$ that do not have a Hamilton cycle: $K_{m,m+1}$; or two copies of $K_m$ with one vertex of each identified. There are many extensions. The first was by Ore: if the degree-sum $d(u) + d(v) \geq n$ for all nonadjacent vertices $u$ and $v$, then the graph has a Hamilton cycle.

**Chvátal–Erdős.** If $G$ is a graph with order at least 3 and the connectivity is at least the independence number, then the graph has a Hamilton cycle.

**Proof.** If the independence number is 1, then the graph is complete. So assume the independence number $k \geq 2$. In particular this means the connectivity is at least 2. Consider the longest cycle $C$. Suppose there exists a vertex $w$ outside the cycle. By Menger’s Theorem, there are internally disjoint paths $P_1, \ldots, P_k$ from $w$ to $C$. If any two paths meet $C$ at consecutive vertices of $C$, then one can insert $w$ to get a longer cycle. So assume not.

For each path $P_t$, let $v_t$ be the vertex of $C$ after the end of $P_t$ going clockwise. We claim that: the set $\{v_t\}$ is an independent set. This follows because a chord $v_tv_{t'}$ would enable a longer cycle using the chord, $P_t$, and $P_{t'}$. 34
But by the condition of theorem, adding $w$ to \{\(v_\ell\)\} yields a set that is not independent. This means that $w$ is adjacent to some \(v_\ell\), and one can thus insert $w$ using \(P_\ell\) and this edge. This contradicts the claim that $C$ is a longest cycle. \(\blacksquare\)

For planar graphs, Tutte showed that 4-connected implies the existence of a Hamilton cycle. In general, a necessary condition for a Hamilton cycle is that the removal of any vertex set $S$ leaves at most $|S|$ components. The **toughness** of a graph is defined to be the minimum of the ratio $|S|/k(G - S)$ over all cut-sets $S$, where $k(G - S)$ is the number of components of $G - S$.

**Open Question.** Does a sufficiently large toughness guarantee a Hamilton cycle?

A graph is **pancyclic** if it contains cycles of each length from 3 up to its order. The minimum-degree threshold to guarantee a graph is pancyclic is only slightly larger than Dirac’s bound:

**Minimum Degree.** If $G$ is a graph with order $n \geq 3$ and minimum degree $\delta \geq (n + 1)/2$, then $G$ is pancyclic.

However, if the graph is known to have a Hamilton cycle, then the threshold can be reduced:

**BFG.** If $G$ is a graph that contains a Hamilton cycle and a triangle and minimum degree $\delta \geq (n + 2)/3$, then $G$ is pancyclic.