Math 9540 — Goddard — Spring25

Assignment 1

1. Consider a graph G created in the following way. There is a set of lines in the plane, no three of which have a common point. Graph G has vertex set the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Show that $\chi(G) \leq 3$. (West 5.1.22)

2. The goal of this question is to improve on the class result.

   (a) Do/redo question 3b of assignment 5 from last year's 8540. That is, show that every 3-regular graph has a $(2,1)$-coloring.

   (b) Define an orange set as a collection of degree-3 vertices S such that S induces a connected subgraph and every vertex in $N(S) - S$ has degree 2. Show that if S is orange, any $(2,1)$-coloring of $G - N[S]$ can be extended to one of G.

   (c) Show that if a graph has minimum degree 2, no adjacent pair of degree-2 vertices, and no orange set, then its average degree is at least 8/3. (Hint: In the discharging, send $\frac{1}{3}$ from each vertex to every neighbor with smaller degree. Also, send $\frac{1}{3}$ between some degree-3 vertices.)

   (d) Using the above, prove that if a graph G has $mad(G) < 8/3$, then it has a $(2,1)$-coloring.

3. Construct a 6-regular graph on 2025 vertices that is embeddable on the torus.

4. (a) Provide an upper bound on the number of edges in a bipartite graph on a surface.
   (b) Show that the hypercube Q_4 has genus 1.
   (c) Show that the hypercube Q_5 has genus at least 5.

Due: Sunday January 26