Coloring Graphs to Produce Walks Without Forbidden Repeats
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Abstract
We consider the problem of coloring the edges of a graph such that every pair of vertices are joined by an $\ell$-rainbow walk, that is, a walk where for every sub-walk of length at most $\ell + 1$ its edges receive different colors. We show that the minimum number of colors needed is at most $2\ell + 1$ if the graph has a cycle of length at least $\ell$. We also provide a sharp bound for $\ell = 2$ in bridgeless graphs, and general bounds for sufficiently large bridgeless graphs, and show that if the graph contains two sufficiently long edge-disjoint closed trails then the number of colors needed is at most $\ell + 2$.

1 Introduction

There is a growing literature on coloring the edges of a graph so that there exist “connectedness substructures” without some forbidden pattern of colors. One version of the problem was introduced by Borozan et al. [2], who defined a coloring where every pair of vertices are joined by a path such that no two consecutive edges on the path have the same color. The associated parameter is called the path connection number. We considered the same coloring problem for walks [14] and for trails [9].

In this paper we consider a related coloring condition. Specifically, we define a walk to be $\ell$-\textit{rainbow} if for every sub-walk of length at most $\ell + 1$, its edges receive different colors. That is, a color can only be re-used every $\ell + 1$\textsuperscript{st} edge. We will denote the condition by $R_\ell$. Thus the original problem was about $R_1$. We define the $R_\ell$-\textit{walk connection number} as the minimum number of colors in a coloring of the edges of graph $G$ such that every pair of vertices are joined by an $R_\ell$-walk, and denote the parameter by $w_\ell(G)$.

This concept for paths was introduced by Li et al. [13] and Chartrand et al. [3, 6] (though the latter define it slightly differently). For example, Li et

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al. [13] showed that if a graph is 2-connected, then there is a coloring of the edges with at most 5 colors such that every pair of vertices are joined by an $R_2$-path. They also provided some calculations, including determining the value of the associated parameter for a complete bipartite graph, which is also given in [3]. The concept was renamed $(1, \ell)$-rainbow connectivity in the survey [11] and book [12], where they denote the associated parameter by $rc_{1,\ell}(G)$. Further, related questions about rainbow-colored paths and cycles were investigated by Chartrand et al. [4, 5]. Related work is considered in [13, 8, 15]. Note that there is also work on rainbow Hamilton cycles, but these are questions about all colorings rather than the existence of a coloring. And there is also work on graphs where the edges are already colored.

Our focus is on walks and in particular upper bounds on the parameter. The original paper contained the bounds for the proper-walk connection number (the 1-walk connection number in the terminology of this paper):

**Theorem 1** [14]

(a) If graph $G$ is connected and not a tree, then $w_1(G) \leq 3$.
(b) If graph $G$ is 2-edge-connected, then $w_1(G) \leq 2$.
And these results are sharp.

An important paper on the walks case is the recent work by Bang-Jensen et al. [1], who gave an algorithmic characterization to determine which graphs have $R_1$-walk connection number 3.

We proceed as follows. In Section 2 we show that the $R_\ell$-walk connection number is at most $2\ell + 1$ if the graph has a closed trail of length at least $\ell$, and discuss the sharpness of this result and its extension. For bridgeless graphs, we show in Section 3 that the $R_2$-walk connection number is at most 4 and this is sharp, and in Section 4 that in general the $R_\ell$-walk connection number is at most $(3\ell + 5)/2$ if the graph is sufficiently large. In Section 5 we consider graphs with two disjoint long closed trails and show that for such graphs, the $R_\ell$-walk number at most $\ell + 2$. Finally in Section 6 we briefly consider some related problems.
2 Walks and Closed Trails

It is immediate that in a tree one cannot reverse a colored walk. Thus the $R_\ell$-walk and $R_\ell$-path connection numbers both equal the version of the edge-chromatic number where edges within distance $\ell$ must have different colors. For $R_2$, Li et al. [13] and Chartrand et al. [3] noted that this value equals one less than the maximum sum of degrees of adjacent vertices. The observation about trees carries over to graphs where every cycle is small. For example, define a pineapple by taking a cycle $C_m$ and adding many leafs at one vertex, as shown in Figure 1. If $m < \ell$, then the walk from one end-vertex to another is short enough that both (and hence all) the leaf-edges must have different colors.

On the other hand, if there is a sufficiently long cycle, then the $R_\ell$-walk connection number is bounded, as we now show. We will need bounds on the (vertex) chromatic number of the powers of cycles (a subset of the so-called Harary graphs). The exact values were given by Kazemi and Jalilolghadr [10]. Let $C_n^\ell$ denote the graph obtained from the cycle $C_n$ by joining all pairs of vertices at distance at most $\ell$.

**Theorem 2** [10] For $1 \leq \ell \leq n/2$, the chromatic number of $C_n^\ell$ is

$$\chi(C_n^\ell) = \left\lceil \frac{n}{n/((\ell + 1))} \right\rceil$$
From this one can deduce the following:

**Lemma 1** Let $2 \leq \ell \leq n/2$.

(a) $\chi(C_n^\ell) \leq 2\ell + 1$.
(b) $\chi(C_n^\ell) = 2\ell + 1$ exactly when $n = 2\ell + 1$.
(c) $\chi(C_n^\ell) = 2\ell$ exactly when (i) $\ell = 2$ and $n$ is not a multiple of 3; (ii) $\ell = 3$ and $n \in \{6, 11\}$; and (iii) $\ell \geq 4$ and $n = 2\ell$.
(d) For fixed $\ell$, $\chi(C_n^\ell) \leq \ell + 2$ for all $n$ sufficiently large.

**Proof.** One can readily prove these claims from scratch. But they also follow from Theorem 2. Say that $n = q(\ell + 1) + r$ where $0 \leq r \leq \ell$. Then we have that

$$\chi(C_n^\ell) = \ell + 1 + \left\lceil \frac{r}{q} \right\rceil.$$

Part (a) follows since $r \leq \ell$. Part (b) follows since the only way that $[r/q]$ can equal $\ell$ is that $r = \ell$ and $q = 1$. Part (d) is immediate as $q \geq \ell$ suffices.

For part (c) assume $\ell = 2$. Then $[r/q] = 1$ when $r \neq 0$ (except for the case covered by (b)). Assume $\ell = 3$. Then $r/q > 1$ when $q = 1$ and $r = 2$ and when $q = 2$ and $r = 3$. For $\ell \geq 4$, the only way that $r/q > \ell - 2$ is that $q = 1$. QED

The above result suggests considering cycles in the graph. But it is sufficient to consider closed trails: that is, closed walks all of whose edges are distinct.

**Theorem 3** If a connected graph $G$ contains a closed trail of length $m$ where $m \geq \ell$, then $w_\ell(G) \leq \ell + [\chi(C_m^\ell)/2] + 1$.

**Proof.** Let $T$ be a closed trail of length $m$. Let $F$ be a forest connecting $T$ to the vertices not in $T$. We will color the edges of $T$ and $F$ such that between any two vertices $u$ and $v$ of $G$, the desired $R_\ell$-walk can be obtained as follows. Let $t_u$ be the vertex of $T$ closest to $u$ and $t_v$ be the vertex of $T$ closest to $v$, where possibly $t_u = u$, $t_v = v$, and/or $t_u = t_v$. The walk proceeds in $F$ from $u$ to $t_u$, traverses $T$ completely back to $t_u$, then walks along $T$ to $t_v$, and then proceeds in $F$ to $v$.

First, we color the closed trail $T$ such that every pair of edges up to distance $\ell$ apart (as measured in their appearance in $T$ with wrap around) receive different
colors. The number of colors needed equals the (ordinary vertex) chromatic number of the $\ell$th power of $C_m$. If $\ell > m/2$ then $\chi(C_m^\ell) = m < 2\ell$. Otherwise, by Lemma 1, $\chi(C_m^\ell) \leq 2\ell + 1$. In particular this implies that $\lceil \chi(C_m^\ell)/2 \rceil \leq \ell + 1$, and since $\lceil x/2 \rceil + \lfloor x/2 \rfloor = x$ for all integers $x$, rearranged this implies that $\chi(C_m^\ell) \leq \ell + \lceil \chi(C_m^\ell)/2 \rceil + 1$. In other words, we have enough colors to color the trail $T$.

We color the edges $e$ of $F$ in order based on their distance $d$ from $T$ (where an edge incident with $T$ has distance 0). There are three cases. If $d \leq \ell - \lceil \chi(C_m^\ell)/2 \rceil$, then we give all edges at distance $d$ the same color, in particular a new color not already used. See Figure 2 for an example for the case $\ell = 3$ and $d = 0$: edge $e_0$ receives the 6th color. The number of new colors needed is at most $\ell - \lceil \chi(C_m^\ell)/2 \rceil + 1$. We have enough colors, since the total number of colors used is $\chi(C_m^\ell) + \ell - \lceil \chi(C_m^\ell)/2 \rceil + 1 = \ell + \lceil \chi(C_m^\ell)/2 \rceil + 1$.

If $\ell \geq d > \ell - \lceil \chi(C_m^\ell)/2 \rceil$, then for edge $e$, let the set $F_e$ consist of the $d$ edges joining $e$ to $T$ together with the $\ell - d$ edges in both directions on $T$. Note that $|F_e| = 2\ell - d$. It suffices to color $e$ so that it does not share a color with any edge in $F_e$. See Figure 2 for an example for the case $\ell = 3$ and $d = 1$: edge $e_1$ can have color 3. Thus we can color edge $e$ greedily provided $\ell + \lceil \chi(C_m^\ell)/2 \rceil + 1 > 2\ell - d$. Rearranged, it is sufficient that $d > \ell - \lceil \chi(C_m^\ell)/2 \rceil$.

Finally, if $d > \ell$, then we let $F_e$ be the first $\ell$ edges on the path joining $e$ to $T$, and give $e$ any color not in $F_e$. QED

As a consequence of Lemma 1a and Theorem 3, we obtain the main result:

![Figure 2: Coloring edges near the trail](image)
**Theorem 4** If a connected graph $G$ contains a closed trail of length at least $\ell$, then $w_\ell(G) \leq 2\ell + 1$.

The above theorem is best possible, as we now show. Define a **whiskered cycle** as the graph obtained from a cycle by attaching at least one end-vertex to every vertex of the cycle. We call the leaf-edges the **whiskers**.

**Lemma 2** Let $G$ be a whiskered $m$-cycle with sufficiently many whiskers at each vertex.

(a) If $m$ is not a multiple of 3, then $w_2(G) = 5$.

(b) If $m \in \{6, 7, 11\}$, then $w_3(G) = 7$.

(c) For $\ell \geq 4$, if $m \in \{2\ell, 2\ell + 1\}$, then $w_\ell(G) = 2\ell + 1$.

**Proof.** Consider a valid coloring. The stated values are upper bounds by the above theorem; so it suffices to show that these many colors are required. Let $C$ denote the cycle. If there are enough whiskers at some vertex $v$ of $C$, then there must be two whiskers at $v$ with the same color. To get between these two end-vertices, one must go around the cycle. Since this condition holds for each choice of vertex $v$, it follows that the cycle $C$ must be colored such that any two edges at distance at most $\ell$ have different colors.

(a) We have $\ell = 2$. By Lemma 1 we know that there are (at least) four colors on $C$. Consider some vertex $v$ of the cycle, and let $e_1, e_2, e_3, e_4$ denote the four consecutive edges in some orientation of $C$ such that $v$ is incident to $e_2$ and $e_3$. Suppose there is a repeated color in the collection $e_1, e_2, e_3, e_4$. Then it must be that $e_1$ and $e_4$ have the same color. But this pattern cannot hold all the way around the cycle $C$, since it would require every 3rd edge to have the same color, and $m$ is not a multiple of 3. It follows that there must be a choice of vertex $v$ where $e_1, e_2, e_3, e_4$ all get different colors. Hence the whiskers incident to $v$ have four forbidden colors; and therefore need a fifth color.

(b,c) We have $\ell \geq 3$. If $m = 2\ell + 1$ then we are done, as one needs $2\ell + 1$ colors for the cycle. If $m = 2\ell$, then it follows that the every whisker needs a color not used on $C$. Since all the edges of $C$ receive different colors, this means that one needs $2\ell + 1$ colors. Finally, consider the case that $\ell = 3$ and $m = 11$. In the coloring of the edges of the cycle $C$, it can be checked by hand or by computer
that for all valid 6-colorings of $C$ there must exist six consecutive edges that receive different colors. Thus one needs a seventh color for the whiskers incident with the middle vertex of that sextet. QED

One can obtain a slightly stronger conclusion for graphs without end-vertices. We define an **Eulerian tour** of graph $G$ as a closed walk that uses every edge of $G$ at least once. If the edges are colored, then an $R_\ell$-**tour** means an Eulerian tour such that every at most $\ell + 1$ consecutive edges have different colors, including wrap around. (That is, the walk can be continued around the tour again and remain $\ell$-rainbow.) Having an $R_\ell$-tour is a stronger condition than having the desired walks.

**Theorem 5** Let $G$ be a connected graph with minimum degree at least 2 that contains a closed trail of length at least $\ell$. Then there is an $(2\ell + 1)$-edge-coloring of $G$ such that there is an $R_\ell$-tour of $G$.

**Proof.** Let $T$ be the closed trail of length at least $\ell$. Define $G'$ as in the proof of Theorem 3 to be a spanning subgraph with the edges of $T$ and a forest connecting every other vertex to the trail. Color $G'$ as in that proof. Choose an orientation of $T$.

Consider some edge $e$ in $G$ but not $G'$. There are three cases: (i) Assume $e = xy$ where both $x$ and $y$ are on $T$. Then let $\mathcal{F}_e$ consist of the $\ell$ consecutive edges of $T$ immediately after (some occurrence of) $y$ together with the $\ell$ consecutive edges of $T$ immediately before (some occurrence of) $x$. (These may overlap.) Give $e$ any color not used on $\mathcal{F}_e$.

(ii) Assume $e = xv$ where $x$ is on $T$ but $v$ is not. Then let $\mathcal{F}_e$ consist of the $\ell$ consecutive edges starting in the forest at $v$ and going onto $T$, together with the $\ell$ consecutive edges of $T$ immediately before (some occurrence of) $x$. Give $e$ any color not used on $\mathcal{F}_e$.

(iii) Assume $e = uv$ where neither $u$ nor $v$ is on $T$. Then let $\mathcal{F}_e$ consist of the $\ell$ consecutive edges starting in the forest at $v$ and going onto $T$, together with the $\ell$ consecutive edges starting in the forest at $u$ and going to $T$ but reverse along $T$. Give $e$ any color not used on $\mathcal{F}_e$. 

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We claim this coloring yields the desired \( R_\ell \)-tour. Our strategy is to start on the closed trail and walk around it and pick up every edge in turn. To pick up an edge of type (i), one can just traverse it and keep walking around \( T \). To pick up an edge of type (ii), exit \( T \) at (the chosen occurrence of) \( x \), traverse the edge, and then walk back down to \( T \) and continue circulating. To pick up an edge of type (iii), exit \( T \) along the forest-path up to \( u \), then traverse the edge \( uv \) and then walk back down to \( T \).

Finally, we note that the above process ensures that each forest edge \( f \) (that is, an edge of \( G' \) not in \( T \)) is traversed. To see this, let \( u \) be an end-vertex of \( G' \) such that the path in \( G' \) from \( u \) to \( T \) contains edge \( f \). Then, since the graph \( G \) has minimum degree at least 2, the vertex \( u \) has an edge of type (ii) or (iii), and \( f \) is traversed when that edge is picked up. \( \text{QED} \)

Of course the condition on the minimum degree is necessary for an \( R_\ell \)-tour. Further, cycles of length at most \( \ell \) do not suffice. To see this, consider for example the barbell consisting of two disjoint such cycles and a path joining them.

### 3 The \( R_2 \)-Walk Connection Number in Bridgeless Graphs

Theorem 4 shows that the \( R_2 \)-walk connection number of a cyclic connected graph is at most 5. We show here that if the graph is bridgeless then the maximum is actually 4.

We need a new idea. Consider a subgraph \( Q \). We say that an Eulerian tour \( W_Q \) of \( Q \) is an \( \ell \)-\textbf{reversible tour} if for every vertex \( v \) in \( Q \) there exists a sub-trail \( S_v \) of length \( \ell \) starting at \( v \) such that \( W_Q \) also includes the reverse of \( S_v \). Note that an \( R_\ell \)-coloring of \( W_Q \) requires at least \( \ell + 1 \) colors, such \( W_Q \) must have at least this many edges.

The first result provides the fundamental connection between reversible tours and walk colorings that “improves” on Theorem 3

**Theorem 6** Let \( G \) be a connected graph. Assume \( G \) has a subgraph \( Q \) with an \( \ell \)-reversible tour \( W_Q \) with an \( R_\ell \)-coloring using \( k \) colors. Then \( w_\ell(G) \leq k \).
Proof. Consider the coloring of $Q$. We noted above that $k \geq \ell + 1$. Let $F$ be a forest such that every vertex not on $Q$ is joined to $Q$ by a path in $F$. For each vertex $v$ of $Q$, let $S_v$ be the sub-trail of $W_Q$ of length $\ell$ starting at $v$. We color the edges of $F$ in order of their distance from $Q$. For each such edge $e$, let $v_e$ be the closest vertex of $Q$. Then consider a walk from $e$ to $Q$ and then along $S_{v_e}$. Define $F_e$ as the first $\ell$ edges of this walk. Give $e$ any color that does not appear in $F_e$.

We claim that this coloring provides the desired $R_\ell$-walks. Suppose that one wants to walk from vertex $x$ to vertex $y$. Let $v_x$ and $v_y$ denote the closest vertices of $Q$. Start at $x$ and walk to $Q$ (if necessary) using the edges of $F$; then proceed along $S_{v_x}$ and continue along $W_Q$ until it traverses $S_{v_y}$ in reverse; and then (if necessary) walk to $y$ using the edges of $F$. \(\text{QED}\)

One natural idea for a suitable subgraph $Q$ is a barbell: two edge-disjoint cycles joined by a (possibly trivial) path. Indeed, two edge-disjoint closed trails are just as good. This approach for $R_1$ was used in the original paper, where it was noted that one needs cycles of suitable parity:

**Lemma 3** [14] If a connected graph $G$ has two edge-disjoint odd cycles, then $G$ has a barbell $Q$ that can be 2-colored to provide a 1-reversible tour.

We consider now the idea for $R_2$. We will need the following well-known fact (given for example as Fact 6 of section 4.8.1 of [16]):

**Lemma 4** If $x$ and $y$ are positive integers with gcd 1, then every positive integer at least $(x - 1)(y - 1)$ can be expressed as a nonnegative integer combination of $x$ and $y$.

**Lemma 5** Let $G$ be a connected graph that contains two edge-disjoint closed trails each of length at least 4. Then $G$ contains a subgraph $Q$ that has an 2-reversible tour with an $R_2$-coloring using 4 colors.

Proof. Let $Q$ be the subgraph obtained by taking the two edge-disjoint closed trails, say $C_1$ and $C_2$, and some path $P$ from $v_1$ to $v_2$ that joins them (possibly $v_1 = v_2$).
We will color each closed trail $C_i$ such that the two edges incident with $v_i$ receive the same color and there is a $R_2$-walk from $v_i$ to $v_i$ around $C_i$. Say $C_i$ has $m$ edges. Then one way to proceed is to use some combination of the patterns $a, b, c$ and $a, b, c, d$ to color the first $m - 1$ edges and then use $a$ for the last edge. By Lemma 4 this approach works except when $m - 1 = 5$. But then the coloring $a, b, c, d, b, a$ works.

First color $P$ arbitrarily so that one has an $R_2$-walk from one end to the other. We need to synchronize the colorings of the $C_i$ with $P$. For each $i$, define $g_i$ as the color that the two edges incident with $v_i$ will receive. We first determine the colors $g_1$ and $g_2$. If $P$ has at least two edges, then choose $g_1$ to be different from the colors of the first two edges of $P$ and $g_2$ to be different from the colors of the last two edges of $P$. If $P$ has only one edge, then choose $g_1$ and $g_2$ to be different from each other and from the color on $P$. And if $P$ has no edge, then simply choose $g_1$ and $g_2$ to be different.

Second, let $p_i$ be the color of the edge of $P$ incident with $v_i$ if $P$ has an edge; and let $p_i$ be $g_{3-i}$ otherwise. Then for each $i$, choose the color(s) of the second and penultimate edge of $C_i$ to be different from $p_i$. See Figure 3 for an example of such a coloring.

This gives one the desired coloring. For a 2-reversible tour of $Q$ with $R_2$-coloring, start by walking along $P$, go around $C_2$ in one direction, go back along $P$, go around $C_1$ in one direction, then again along $P$ but then traverse $C_1$ and $C_2$ in the other direction. QED

What do graphs look like that don’t have two edge-disjoint cycles? For exam-
ple, Erdős and Pósa [7] noted that the number of edges is at most 3 more than the order. This suggests the following fact is probably known:

**Lemma 6** Let $G$ be a 2-connected graph that does not contain two edge-disjoint cycles. Then $G$ has a spanning subgraph that is either a Hamilton path or a subdivision of $K_4$ or $K_{3,3}$.

**Proof.** Consider a cycle $H_1$. If this contains all vertices of $G$, we are done. So assume there is a vertex not on $H_1$. This vertex has two internally-disjoint paths to $H_1$, and so one obtains a subgraph $H_2$ of $G$ that is a subdivision of $K_4 - e$.

If $H_2$ contains all vertices of $G$, then it contains a Hamilton path and we are done. So assume there is a vertex not on $H_2$. Then the vertex has two internally disjoint paths to the subgraph $H_2$. If these paths use either of the degree-3 vertices of $H_2$, then it is readily checked that there are two edge-disjoint cycles. So these paths do not use these vertices, and one has a subgraph $H_3$ of $G$ that is a subdivision of $K_4$.

If $H_3$ is all of $G$ then we are done. So assume there is some vertex outside $H_3$. Then this vertex has two internally disjoint paths to $H_3$. This yields two edge-disjoint cycles except in the case that these paths meet $H_3$ at degree-2 vertices that are on paths that don’t share an endpoint. This yields a subdivision of $K_{3,3}$.

It can then be checked that the existence of a vertex outside the $K_{3,3}$ creates a second cycle. \(\text{QED}\)

As a consequence we obtain the following result:

**Theorem 7** If a graph $G$ is 2-edge-connected, then $w_2(G) \leq 4$.

**Proof.** If the graph has a Hamilton path, then three colors are needed to color the path to provide the desired walks. If the graph has a triangle, then the upper bound follows from Theorem 3. If the graph has two edge-disjoint cycles, then the upper bound follows from Theorem 6 and Lemma 5. So assume the graph has none of this. In particular, this means that there is no cut-vertex. By the above lemma, the graph is a subdivision of $K_4$ or $K_{3,3}$.
Consider first the case that $G$ is a subdivision of $K_4$. Let $C$ be a cycle that contains all four vertices of degree 3, say $v_1, v_2, v_3, v_4$. Suppose that $C$ is a 5-cycle, say with $v_1, v_2, v_3, v_4$ as subpath. Then by the lack of triangles, both the path from $v_1$ to $v_3$ outside $C$ and the path from $v_2$ to $v_4$ outside $C$ must contain at least one internal vertex. Thus it is possible to choose $C$ to not be a 5-cycle.

Then let $P_1$ be the path from $v_3$ to $v_1$ without the first vertex, and let $P_2$ be the path from $v_4$ to $v_2$ without the first vertex (so that a spanning subgraph consisting of $C$ with $P_1$ and $P_2$ dangling off it). Color the cycle $C$ with four colors so that one can circulate around it. (This is possible by Lemma 1.) For each path $P_i$, color it such that one can walk along it and join $C$ and go around $C$ the long way round to the other path (that is, using $v_3$ and $v_4$). This allows one to go down $P_1$ and up $P_2$. Thus this is a valid $R_2$-coloring.

Consider second the case that $G$ is a subdivision of $K_{3,3}$. Let $C$ be a cycle that contains all six vertices of degree 3, say meeting them in the order $v_1, \ldots, v_6$. Then for $i = 1, 2, 3$ let $P_i$ be the path from $v_{3+i}$ to $v_i$ without the first vertex. If the set $\{v_1, \ldots, v_6\}$ is not independent, then one can choose $C$ such that $P_3$ has no edge. Then one can proceed as in the case that $G$ is a subdivision of $K_4$. So assume that none of the $v_i$ are adjacent. It follows that $C$ has length at least 12. Color the cycle $C$ with four colors so that one can circulate around it. (Possible by Lemma 1.)

Let $e_1, e_2, e_3, e_4$ be the two edges before and after $v_2$ on $C$ (that is, $v_2$ is incident to $e_2$ and $e_3$). We claim that one can choose the coloring of $C$ such that $e_1$ and $e_4$ have the same color; for if every choice of $e_1$ and $e_4$ have different colors, then the coloring of $C$ must look like $1, 2, 3, 4, 1, 2, 3, 4, \ldots$ Since $C$ is long enough we can choose a coloring that does not have the property that every fifth edge has the same color. (For example, for the case of length 12 one can use four copies of the pattern $1, 2, 3$.)

Color $P_2$ as follows. Give the edge of $P_2$ incident to $v_2$ the color not present on any of the edges $e_i$, give the next edge of $P_2$ the color of $e_1$ and $e_4$, and then proceed along $P_2$. This ensures that one can walk down $P_2$ and join $C$ and go either way around $C$. For paths $P_1$ and $P_3$, color them so that one can walk along down $P_1$ go around $C$ in some direction and then go up $P_3$. This also provides the walk from $P_1$ or $P_3$ to $P_2$. QED
3.1 Rosary graphs

The value 4 in Theorem 7 is best possible, as we now show. Define a **rosary graph** as follows. Take three cycles $C_1, C_2, C_3$ each of length $2a$. For each cycle, choose one pair of antipodal vertices $d_i, d'_i$ ($i = 1, 2, 3$). Then join $d'_i$ to $d_{i+1}$ (arithmetic modulo 3) with a path $P_i$ that has at least one internal vertex. Let $b$ denote the total number of vertices added. An example with $a = 5$ and $b = 4$ is shown in Figure 4.

**Lemma 7** If $a \geq 5$, and neither $a$ nor $b$ is a multiple of 3, then the rosary graph has $w_2 \geq 4$.

**Proof.** Suppose there is a valid 3-coloring of the edges. The condition on $a$ and $b$ means that none of the cycles in the graph has length a multiple of 3. Note that every $R_2$-walk must be periodic, that is, colored $1, 2, 3, 1, 2, 3, \ldots$

Since $a \geq 5$, there exists a pair of antipodes $c_1, c'_1$ of $C_1$ that are not adjacent to $d_1, d'_1$. Let $x$ be the penultimate vertex on $P_3$, and $y$ the second vertex on $P_1$. We claim that for some choice of $\alpha \in \{c_1, c'_1\}$ and $\beta \in \{x, y\}$, there is an $R_2$-walk
between $\alpha$ and $\beta$ going the long way round, meaning using each of $P_1$ through $P_3$. This is immediate if the $R_2$-walk between $c_1$ and $c'_1$ goes the long way round, as it then must use both $x$ and $y$.

So assume without loss of generality that the $R_2$-walk from $c_1$ to $c'_1$ stays within the first cycle $C_1$ and uses $d_1$. Consider the two edges before and after $d_1$ on the walk. Since the edge $d_1x$ shares a color with one of these edges, in the graph the $R_2$-walk from one of $c_1, c'_1$, say $c_1$, cannot get onto $P_3$ there. Assume the $R_2$-walk from $c_1$ to $x$ uses only the first cycle. Then the walk $c_1$ to $d'_1$ to $c'_1$ to $d_1$ to $c_1$ is $R_2$-colored and hence must be periodic. But since $a$ is not a multiple of 3 and $a \geq 5$, there is a choice of antipodes and so one can choose vertex $c_1$ that does not have this property. It follows that we may assume the $R_2$-walk from $c_1$ to $x$ goes the long way around.

That is, there is a $R_2$-walk from $C_1$ to $C_1$ using all the $P_i$ and half of each of the other cycles. The same argument holds for each cycle $C_2$ and $C_3$ in turn. These colorings overlap and so give a proper $R_2$-coloring of a long cycle. But this presents a contradiction, since the long cycle has length $b + 3a$, which is not a multiple of 3. \textit{QED}

4 General Bridgeless Graphs

The approach of Section 2 can provide an upper bound for graphs with long closed trails.

We show next that 2-edge-connected graphs whose longest closed trail is bounded have bounded order. It is probable that this fact has been noted before, and maybe with a realistic upper bound.

\textbf{Lemma 8} For all $t$, there are finitely many 2-edge-connected graphs whose maximum closed trail length is at most $t$.

\textbf{Proof.} The proof is by induction on $t$. The base case is immediate; for example for $t = 3$ the only graph is $K_3$. So consider a longest closed trail $T$ in graph $G$.

We claim that the longest closed trail in $G - E[T]$ has length less than $t$. Consider a longest closed trail $T'$ that is edge-disjoint from $T$; say of length $t'$.
By the maximality of $T$, the trails $T$ and $T'$ have no vertex in common. By the connectivity of $G$, there are two edge-disjoint paths from $T'$ to $T$. These two paths can be combined with at least half of each of $T$ and $T'$ to get a closed trail in $G$ of length at least $t/2 + t'/2 + 2$. It follows that $t' \leq t - 4$, which proves the claim.

Now, for each vertex $v$ outside $T$, there exist two edge-disjoint paths to $T$; choose one such pair and call this the $v$-splay. The $v$-splay must end in different vertices of $T$, else we have a contradiction of the maximality of $T$. Further, these ends cannot be consecutive on $T$, else we again get a contradiction. For each pair $P$ of vertices of $V(T)$ that are not joined by an edge of $T$, define the auxiliary subgraph $H_P$ that consists of all $v$-splays with that pair of terminals, and adding the edge $e_P$ joining $P$ (if necessary).

Assume the graph $H_P$ is not $K_2$. Then the removal of a single edge cannot disconnect $H_P$, because the two vertices of $P$ are still joined (either by their edge or by a splay), and every other vertex can get to at least one vertex in $P$. That is, $H_P$ is 2-edge-connected. Consider a closed trail $Q$ in $H_P$. If $Q$ does not use $e_P$, then $Q$ is in $G - E[T]$ and so has length less than $t$ by the above claim. If $Q$ does use $e_P$, then let $Q'$ be the closed trail in $G$ that is formed by replacing the edge $e_P$ by a segment of $T$. It follows that the length of $Q'$ is more than the length of $Q$, while $Q'$ has at most $t$ edges. Hence the longest closed trail in $H_P$ has length at most $t - 1$.

By the induction hypothesis, it thus follows that the graph $H_P$ has bounded order. Every vertex of $G$ is in at least one of the auxiliary graphs $H_P$. Since there are $(t - 1)(t - 2)/2$ choices for $P$, it follows that $G$ has bounded order. QED

And so we get the following result:

**Theorem 8** If a graph $G$ is 2-edge-connected and sufficiently large, then $\omega_t(G) \leq \ell + 2 + \lceil \ell/2 \rceil$.

**Proof.** By Lemma 1, there is a $t_0$ such that $\chi(C_t^\ell) \leq \ell + 2$ for all $t \geq t_0$. By Lemma 8, if $G$ is sufficiently large it contains a closed trail of length at least $t_0$. By Theorem 3 the bound follows. QED
We do not know if this is even close to best possible, neither the value, nor whether the requirement that $G$ be sufficiently large is necessary. It does seem likely that there is a suitable version $R_\ell$ of a rosary graph where $w_\ell(R_\ell) \geq (1+\epsilon)\ell$, but we are unable to prove this.

5 Two Disjoint Closed Trails

We saw above that having a long closed trail improves the upper bound on the $R_\ell$-walk number from $2\ell + 1$ to $3\ell/2 + 5/2$. We show next that having two long edge-disjoint closed trails improves the upper bound to $\ell + 2$. We will need the following lemma.

Lemma 9 Fix some $\ell$. If $n$ is sufficiently large, then the path on $n$ edges has an $R_\ell$-walk coloring using $\ell + 2$ colors such that the first $\ell - 1$ edges and the last $\ell - 1$ edges have the same colors but reversed.

Proof. Let $F_0$ denote the pattern of colors $1, 2, \ldots, \ell - 1$. Let $X$ denote the other three colors (that is, $\ell, \ell + 1, \ell + 2$). We start with a color scheme of the form $F_0, X, F_1, X, F_2, X, \ldots$, where each $F_i$ is a permutation of the colors from 1 up to $\ell - 1$. The key is that, for each pair of consecutive $F_i$'s, the colors are rearranged slightly, with each color moving at most one position. So, for example, by doing what computer science calls a bubble sort, after sufficiently many steps, the ordering is reversed. For instance, if $\ell = 4$ the coloring might start

\[1 \ 2 \ 3 \ x \ x \ x \ 1 \ 3 \ 2 \ x \ x \ x \ 3 \ 1 \ 2 \ x \ x \ x \ 3 \ 2 \ 1\]

Once the $F_0$ has been reversed to $F_0^R$, then if sufficiently long one can by Lemma 4 write the remaining number of edges as an integer combination of $\ell + 1$ and $\ell + 2$. Then the remaining edges of the path are colored with some combination of the patterns $X F_0^R$ or $X' F_0^R$ where $X'$ is $\ell, \ell + 1$ say. QED

Theorem 9 If connected graph $G$ has two sufficiently long edge-disjoint closed trails, then it has a subgraph $Q$ that has an $\ell$-reversible tour with an $R_\ell$-coloring using $\ell + 2$ colors.
**Proof.** Say the two edge-disjoint closed trails are $C_1$ and $C_2$. Let $P$ be a path with ends $v_1$ to $v_2$ that joins $C_1$ and $C_2$ (possibly $v_1 = v_2$). Let $Q$ be the subgraph with the edges of $P$, $C_1$, and $C_2$ and the incident vertices. The goal is to color $Q$ such that it is possible to start on $P$ and walk along, go around the closed trail $C_1$ in either direction, go back along $P$, around the closed trail $C_2$ in either direction, and so on. By the above lemma, one can give the cycles a suitable coloring. Further, one can choose the coloring of the cycles and the path such that the $\ell - 1$ edges of $C_1$ ending at $v_1$, followed by $P$, followed by the $\ell - 1$ edges of $C_2$ starting at $v_2$, form an $\ell$-rainbow walk. QED

It seems possible that every sufficiently large 3-connected graph has two arbitrarily long disjoint closed trails. This would imply that every sufficiently large 3-connected graph has $R_\ell$-walk number at most $\ell + 2$.

**6 Other Comments**

We have so far looked at the question of strengthening the $R_1$ condition. In another direction, one could for example consider the problem where the color cannot be the same for three consecutive steps. But here two colors is enough for path connection: simply take a spanning tree, designate a root, and color the edges of the tree based on the parity of their distance from the root.

For future work, one could consider stronger restrictions, such as that every two vertices are in a cycle, or there are multiple “disjoint” walks/trails/paths between them, as for example investigated in [8, 13]. One can also consider an upper bound on the length of the walk, as was done for paths in [15].

The complexity question for $w_\ell$ for $\ell \geq 2$ is open.

**References**


