Construction of Trees and Graphs with Equal Domination Parameters

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Abstract

We provide a simple constructive characterization for trees with equal domination and independent domination numbers, and for trees with equal domination and total domination numbers. We also consider a general framework for constructive characterizations for other equality problems.

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1 Introduction

For any two graph parameters λ and μ, we define a graph G to be a (λ, μ)-graph if $\lambda(G) = \mu(G)$. Several papers have considered the problem of characterizing when two related domination parameters of a graph are equal. These include [3, 6, 7]. See also [8, Section 3.5.2].

We will need the following definitions. Let $G = (V, E)$ be a simple undirected graph. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to a vertex of S; the *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set (or equivalently, the minimum cardinality of a maximal independent set). The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a dominating set where every vertex in the set also has a neighbor in the set. The set S is a *packing* if the vertices in S are pairwise at distance at least 3 apart in G; the *packing number* $\rho(G)$ is the maximum cardinality of a packing. For λ one of these parameters, a λ-set is one where equality is attained. For a survey see [8, 9]. For two graph parameters λ and μ, we write $\lambda \le \mu$ if $\lambda(G) \le \mu(G)$ for all graphs G. For example, $\rho \le \gamma \le \{i, \gamma_t\}$.

It is known that (γ, i)-graphs are difficult to characterize. Several classes of (γ, i)-graphs have been found—see, for example, [1, 2, 4, 5, 14]. The class of (γ, i)-trees was first characterized by Harary and Livingston [6] but this characterization is rather complex. Recently, Cockayne et al. [3] provided a characterization of (γ, i)-trees in terms of the sets $\mathcal{A}(T)$ and $\mathcal{A}_i(T)$ of vertices of the tree T which are contained in all its γ-sets and i-sets, respectively. These sets were characterized by the fourth author [12] using a tree-pruning procedure.

In another direction, Haynes et al. [10] provided a constructive characterization of those trees with strong equality: that is, where every γ-set is an i-set. If instead one requires the graph to be domination perfect (that is, $\gamma(G') = i(G')$ for all subgraphs G' of G), it is easy to show that a tree is domination perfect iff it does not contain two adjacent vertices of degree 3 or more. (This is also a corollary of the results in any of [5, 13, 14].)

In this paper we provide a constructive characterization of (γ, i)-trees that is simpler than those mentioned above. We also provide a constructive characterization of (γ, γ_t)-trees, and show how to generate all (ρ, γ)-, (ρ, i)- and (ρ, γ_t)-graphs.

For notation and graph-theory terminology we in general follow [8]. A *leaf* of a tree T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. For a vertex v in a rooted tree T we denote by T_v the subtree of T induced by v and its descendants. A path of order n we denote by P_n.

We will need the following fact.

Fact 1 (Moon and Meir [11]) *For a tree T, $\gamma(T) = \rho(T)$.*

2 Labelings

The key to our constructive characterization of graphs with equal values of two parameters is to find a labeling of the vertices that indicates the roles each vertex plays in the sets associated with both parameters.

Let λ be a graph parameter. We say that λ is a *max-set* parameter if there exists a property π_λ of subsets of vertices such that $\lambda(G)$ is the maximum cardinality of a π_λ-set of any graph G (and a λ-set is always a π_λ-set). It is a *min-set* parameter if there exists a property σ_λ such that $\lambda(G)$ is the minimum cardinality of a σ_λ-set of G. For example, a π_ρ-set is a packing and a σ_γ-set is a dominating set.

If λ is a max-set parameter and μ a min-set parameter, then we define a (λ, μ)*-labeling* of a graph $G = (V, E)$ as a partition $S = (S_A, S_B, S_C, S_D)$ of V such that $S_A \cup S_D$ is a σ_μ-set, $S_C \cup S_D$ is a π_λ-set, and $|S_A| = |S_C|$.

Lemma 2 *Let λ be a max-set parameter and μ a min-set parameter such that $\lambda \le \mu$. Then a graph is a (λ, μ)-graph if and only if it has a (λ, μ)-labeling.*

Proof. Suppose G has a (λ, μ)-labeling. Then $\mu(G) \le |S_A \cup S_D| = |S_C \cup S_D| \le \lambda(G)$, and so $\mu(G) = \lambda(G)$. Suppose G is a (λ, μ)-graph. Let L be a λ-set and M a μ-set. Then a (λ, μ)-labeling is given by $S_A = M \setminus L$, $S_B = V \setminus (M \cup L)$, $S_C = L \setminus M$ and $S_D = L \cap M$. Since $\lambda(G) = \mu(G)$, it follows that $|S_A| = |S_C|$. □

We will refer to the pair (G, S) as a λ-μ-graph. The *label* or *status* of a vertex v, denoted sta(v), is the letter $x \in \{A, B, C, D\}$ such that $v \in S_x$. A *labeled* graph is simply one where each vertex is labeled with either A, B, C or D.

We will need the following lemma:

Lemma 3 *Consider a (ρ, γ)-labeling. If $v \in S_A$ (resp. S_C), then v is adjacent to exactly one vertex of S_C (resp. S_A), and to no vertex of S_D. If, moreover the labeling is a (ρ, γ_t)-labeling, then $S_D = \emptyset$.*

Proof. Since S_C is a packing, a vertex in S_A is adjacent to at most one vertex in S_C. Every vertex in S_C must be adjacent to at least one vertex in S_A, since it is dominated by $S_A \cup S_D$ and is not adjacent to a vertex in S_D. Since a vertex in S_A can be adjacent to at most one vertex in S_C, and $|S_C| = |S_A|$, a vertex in S_C cannot have two neighbors in S_A (otherwise some other vertex in S_C has no neighbor in S_A), and every vertex in S_A must be adjacent to a vertex in S_C.

In particular, every vertex of S_D has neighbors only in S_B. Thus, if we have a (ρ, γ_t)-labeling, then $S_D = \emptyset$. □

We now define some graph operations.

- **Operation $\mathcal{G}_1$.** Assume $\text{sta}(y) \in \{A, D\}$. Add a vertex x and the edge xy. Let $\text{sta}(x) = B$.
- **Operation $\mathcal{G}_2$.** Assume $\text{sta}(y) = A$ and $\text{sta}(z) = C$. Add a vertex x and the edges xy and xz. Let $\text{sta}(x) = B$.
- **Operation $\mathcal{G}_3$.** Assume $\text{sta}(x), \text{sta}(y) \in \{A, B\}$. Add the edge xy.
- **Operation $\mathcal{G}_4$.** Assume $\text{sta}(y) = A$. Add a path x, w and the edge xy. Let $\text{sta}(x) = A$ and $\text{sta}(w) = C$.

These operations are illustrated in Figure 1.

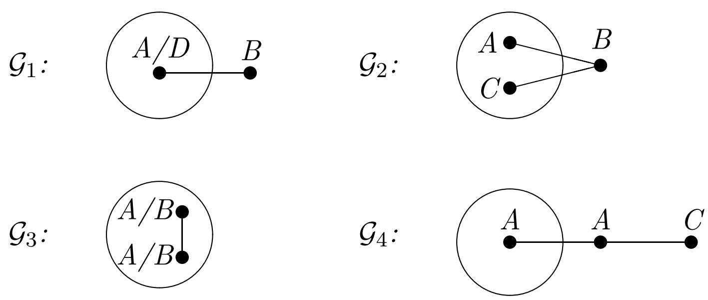

Figure 1: The four $\mathcal{G}_i$ operations

Theorem 4 *A labeled graph is a ρ-γ-graph if and only if it can be obtained from a disjoint union of P_1's, labeled D, and P_2's, labeled A and C, using operations $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.*

Proof. It is clear that the operations produce only the claimed labelings. That is, each of the three operations preserves the property that $S_A \cup S_D$ is a dominating set and $S_C \cup S_D$ is a packing.

For the proof that every such labeling can be produced, we proceed by induction on the sum of the numbers of vertices and edges. For the base case, consider any ρ-γ-graph with every component either a P_1 with vertex labeled D or a P_2 with vertices labeled A and C. Such a labeled graph is produced since the components are supplied and disjoint union is permitted.

Consider the general case for graph H. If there is an edge e inside S_A or inside S_B, induct on $H - e$: that is, the labeled graph $H - e$ is a ρ-γ-graph, and by the induction hypothesis can be produced by the above operations; the edge e can then be restored with operation $\mathcal{G}_3$.

So we may assume S_A and S_B are both independent sets. If there is an edge $e = xy$ with $x \in S_A$ and $y \in S_B$, then one can delete e and induct on $H - e$ as above, unless y is undominated by $S_A \cup S_D$ in $H - e$. In this case, y has at most one other neighbor, namely a vertex $z \in S_C$. So, one can delete y, induct on $H - y$, and restore y with operation $\mathcal{G}_1$ or $\mathcal{G}_2$.

So we may assume that there is no edge joining S_A to S_B. Since every vertex has a neighbor in $S_A \cup S_D$, every vertex y of S_B is a leaf, with a neighbor in S_D. Again one can delete y, induct on $H - y$, and restore y with $\mathcal{G}_1$.

So we may assume that $S_B = \emptyset$. Then the graph H is the disjoint union of P_1's, labeled D, and P_2's, labeled A and C, as in the base case. □

Theorem 5 *A labeled graph is a ρ-i-graph if and only if it can be obtained from a disjoint union of P_1's, labeled D, and P_2's, labeled A and C, using operations $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ as in Theorem 4, but without using the operation $\mathcal{G}_3$ that adds an edge between two vertices with status A.*

Proof. It is clear that the operations produce only the claimed labelings. That is, each of the three operations preserves the property that $S_A \cup S_D$ is an independent dominating set and $S_C \cup S_D$ is a packing.

The proof that every such labeling can be produced, is almost the same as that in Theorem 4. The only difference is that, since $S_A \cup S_D$ is an independent set, there can be no edge between two vertices of S_A and so the operation of $\mathcal{G}_3$ to join two vertices of S_A is not used. □

Theorem 6 *A labeled graph is a ρ-γ_t-graph if and only if it can be obtained from a disjoint union of P_4's, with end-vertices labeled C and internal vertices labeled A, using operations $\mathcal{G}_1$ through $\mathcal{G}_4$.*

Proof. It is clear that the operations produce only the claimed labelings. That is, each of the four operations preserves the property that $S_A \cup S_D$ is a total dominating set and $S_C \cup S_D$ is a packing.

For the proof that every such labeling can be produced, we proceed by induction on the sum of the numbers of vertices and edges. The total domination number of a graph is at least 2. Thus the smallest ρ-γ_t-graph has 4 vertices and is the P_4 provided. This establishes the base case of the induction.

Consider the general case for graph H. If there is an edge b inside S_B, delete e and induct: that is, the graph $H - e$ is a ρ-γ_t-graph, and by the induction hypothesis can be produced by the above operations; the edge e can then be restored with operation $\mathcal{G}_3$.

If there is an edge $e = xy$ with $x \in S_A$ and $y \in S_B$, then one can delete e and induct on $H - e$ as above, unless y is undominated in H_e. In this case, y has at most one other neighbor, namely a vertex $z \in S_C$. So, one can delete y, induct on $H - y$, and restore y with operation $\mathcal{G}_1$ or $\mathcal{G}_2$.

So we may assume that there is no edge joining S_A to S_B. Since every vertex has a neighbor in $S_A \cup S_D$, every vertex y of S_B is a leaf, with a neighbor in S_D. Again one can delete y, induct on $H - y$, and restore y with $\mathcal{G}_1$. So we may assume that $S_B = \emptyset$.

Thus, every vertex of S_C is a leaf. If some component of the induced subgraph $\langle S_A \rangle$ is not a star, then it has an edge e whose removal does not isolate a vertex of S_A. So the graph $H - e$ is a ρ-γ_t-graph, and one can delete e, induct on the graph $H - e$, and restore the edge e using $\mathcal{G}_3$.

So we may assume that $\langle S_A \rangle$ is a union of stars. If $\langle S_A \rangle$ has only components with single edges, then we are done: H is the union of P_4s. Otherwise, there is a component with more than one edge. In this component, let v be a leaf (as viewed in $\langle S_A \rangle$), and let w be its C-neighbor. Consider the graph $H - \{v, w\}$. This is a ρ-γ_t-graph, and so one can induct on $H = \{v, w\}$, and use operation $\mathcal{G}_4$ to restore v and w. □

2.1 Other graph families

One can also characterize or generate ρ-γ-, ρ-i-, or ρ-γ_t-graphs that are *bipartite*. The algorithm is simply to allow only those steps that preserve bipartiteness. One way to ensure this is to keep track of the 2-coloring via a modified labeling—for example, by labeling with A or A' etc. and requiring that each edge joins a vertex with a primed label to one with an unprimed label.

One can similarly construct all labeled *forests* by allowing only those steps that preserve acyclicity. (That is, $\mathcal{G}_2$ and $\mathcal{G}_3$ are permitted only if they do not create a cycle.) However, this construction is unsatisfactory as a way to characterize *trees*, since one cannot use the local labeling to check whether a cycle would be created, and also the intermediate graphs are forests instead of trees. The main result in this paper is a set of operations which produce exactly the ρ-i-trees.

3 Building ρ-i-trees

We now describe a procedure to build ρ-i-trees. Let $\mathcal{L}$ be the minimum family of labeled trees that:

(i) contains (P_1, S_1) where the single vertex has status D, and contains (P_2, S_2) where one vertex has status A and the other status C; and

(ii) is closed under the six operations $\mathcal{T}_j$ ($j = 1, \ldots, 6$) listed below, which extend the tree T by attaching a tree to the vertex $y \in V(T)$, called the *attacher*.

- **Operation $\mathcal{T}_1$.** The same as operation $\mathcal{G}_1$.
- **Operation $\mathcal{T}_2$.** Assume $\mathrm{sta}(y) \in \{A, B\}$. Add a path x, w and the edge xy. Let $\mathrm{sta}(x) = B$ and $\mathrm{sta}(w) = D$.

- **Operation $\mathcal{T}_3$.** Assume $\mathrm{sta}(y) = B$. Add a path x, w and the edge xy. Let $\mathrm{sta}(x) = A$ and $\mathrm{sta}(w) = C$.
- **Operation $\mathcal{T}_4$.** Assume $\mathrm{sta}(y) \in \{B, C\}$. Add a path x, w, z and the edge xy. Let $\mathrm{sta}(x) = B$, $\mathrm{sta}(w) = A$ and $\mathrm{sta}(z) = C$.
- **Operation $\mathcal{T}_5$.** Assume $\mathrm{sta}(y) = A$. Add a path x, w, z and the edge xy. Let $\mathrm{sta}(x) = B$, $\mathrm{sta}(w) = C$, and $\mathrm{sta}(z) = A$.
- **Operation $\mathcal{T}_6$.** Assume $\mathrm{sta}(y) = B$. Add a path v, u, x, w, z and the edge xy. Let $\mathrm{sta}(x) = B$, $\mathrm{sta}(w) = \mathrm{sta}(v) = C$, $\mathrm{sta}(z) = \mathrm{sta}(u) = A$.

These operations are illustrated in Figure 2.

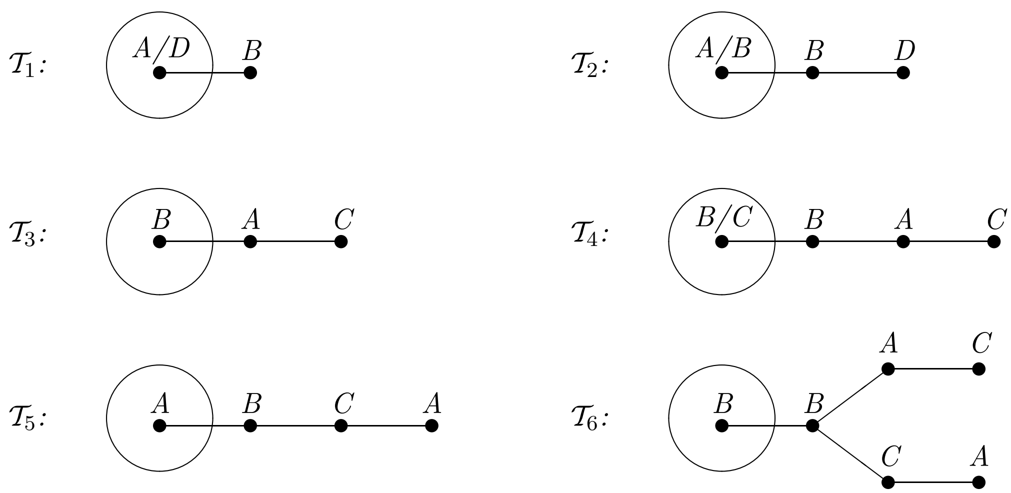

Figure 2: The six $\mathcal{T}_i$ operations

Theorem 7 *A labeled tree is a ρ-i-tree if and only if it is in $\mathcal{L}$.*

Proof. It is easily checked that every element of $\mathcal{L}$ is a ρ-i-tree. That is, each of the six operations preserves the property that $S_A \cup S_D$ is a dominating set and $S_C \cup S_D$ is a packing.

The proof that every ρ-i-tree (T, S) is in $\mathcal{L}$ is by induction on the order of T. For the base case consider any star T. It follows easily that there is a construction of (T, S) for any ρ-i-labeling S by starting with either the P_1 or the P_2 and repeatedly using $\mathcal{T}_1$.

So fix a ρ-i-tree (T, S), and assume that any smaller ρ-i-tree is in $\mathcal{L}$. We may assume that $\mathrm{diam}(T) \geq 3$, since otherwise T is a star, which we have already dealt with.

Let $I = S_A \cup S_D$ and $P = S_C \cup S_D$. We will need the following lemma.

Lemma 8 *Let u be any vertex of T other than the root, with v the parent of u, and let (T', S') be the labeled tree formed by the deletion of T_u. Suppose that (T, S) can be obtained from (T', S') by attaching T_u to v using an operation $\mathcal{T}_j$. Then $(T, S) \in \mathcal{L}$ except possibly if $j = 3$ and v is not dominated by $I \setminus \{u\}$.*

Proof. We want to show that (T', S') is a ρ-i-tree, since then, by the inductive hypothesis, $(T', S') \in \mathcal{L}$, and so can be extended to (T, S) by using the operation $\mathcal{T}_j$.

For any set $Z \subseteq V(T)$ let $Z' = Z \cap V(T')$. For all operations, the number of vertices of T_u of status A equals the number of vertices of T_u of status C, so $|S'_A| = |S'_C|$. Since P is a packing, P' is a packing. Since I is independent, I' is independent. Since I dominates T, I' will dominate T' provided v is dominated by an element of I other than u. If $j = 3$, this is assumed. If $j \neq 3$, then u has status B and so this is necessarily the case. □

We return to the proof of Theorem 7. Consider a longest path $z, y, x, w, \ldots, r$ (possibly $w = r$) and root the tree T at r.

Suppose $\mathrm{sta}(z) = B$. Then since z is dominated by I, the vertex y has status A or D. And so $(T, S) \in \mathcal{L}$ by Lemma 8 with $u = z$ and $j = 1$.

So we may assume that no eccentric vertex has status B. Suppose $\mathrm{sta}(z) = D$. Then by Lemma 3, $\mathrm{sta}(y) = B$. Since P is a packing, any neighbor of y has status A or B. This means that y has no other leaf neighbor (since a vertex with status A has a neighbor with status C) and so has degree 2. Thus $(T, S) \in \mathcal{L}$ by Lemma 8 with $u = y$ and $j = 2$.

So we may assume that every eccentric vertex has status A or C. So, by Lemma 3, every vertex at distance two from an eccentric vertex has status B. In particular, this means that y has degree 2.

Suppose $\mathrm{sta}(z) = C$. Then $(T, S) \in \mathcal{L}$ by Lemma 8 with $u = y$ and $j = 3$, unless x has no neighbor in $I \setminus \{u\}$. So suppose that is the case. Then $\mathrm{sta}(w) \in \{B, C\}$. If x has degree 2, then $(T, S) \in \mathcal{L}$ by Lemma 8 with $u = x$ and $j = 4$. Hence assume $\deg(x) \geq 3$. This means that x has a neighbor $y' \neq w$ that has status B or C. Since I dominates y', the vertex y' has a neighbor z' with $\mathrm{sta}(z') \in \{A, D\}$; clearly z' is eccentric and so (as above) $\deg(y') = 2$. By Lemma 3 and the above assumptions, $\mathrm{sta}(z') = A$ and $\mathrm{sta}(y') = C$. But x can only have one neighbor with status C, and so has degree 3. Thus $(T, S) \in \mathcal{L}$ by Lemma 8 with $u = x$ and $j = 6$.

Hence we may assume that all eccentric vertices have status A. This means that all neighbors of x, apart from w, have status C, and so x has degree 2. It follows that $\mathrm{sta}(w) = A$. Thus $(T, S) \in \mathcal{L}$ by Lemma 8 with $u = x$ and $j = 5$. □

By Fact 1 (in Section 1), it follows that:

Corollary 9 *The (γ, i)-trees are precisely those trees T such that $(T, S) \in \mathcal{L}$ for some labeling S.*

3.1 Minimality of $\mathcal{L}$

We investigate next the question of whether every operation is needed. We will construct a particular labeled tree where the (ρ, i)-labeling is unique up to isomorphism and in which every operation and attacher status is essential.

Let R be the tree obtained from the path u, x, w by adding two leaves z_1, z_2 adjacent to w. Then let $\mathcal{T}_{AB}$ be the operation that attaches a copy of R to a vertex y of status A or B with the edge xy, such that $\text{sta}(x) = B$, $\text{sta}(w) = A$, $\text{sta}(u) = D$ and $\{\text{sta}(z_1), \text{sta}(z_2)\} = \{B, C\}$.

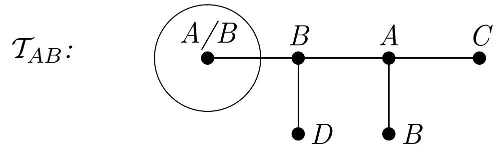

Figure 3: The $\mathcal{T}_{AB}$ operation

For a ρ-i-tree (T, S) we define (T', S') to be the tree obtained by applying $\mathcal{T}_1$ twice to each vertex of T of status A or D, and define (T^*, S^*) to be the tree obtained from (T', S') by applying $\mathcal{T}_{AB}$ to every vertex of T' of status A or B.

The next lemma shows that the ρ-i-labeling of T^* is unique up to isomorphism.

Lemma 10 *Let (T, S) be any ρ-i-tree and let S^* be any ρ-i-labeling of T^*. Then S^* is unique except that the labeling of a pair of leaves at distance two from each other can be swapped.*

Proof. The first claim is that each subgraph added by $\mathcal{T}_{AB}$ must receive its original labeling. (Start by arguing that at least one of z_1 and z_2 receives status B and so w has status A or D. But then look at x and u, etc.) Further, the attacher for a $\mathcal{T}_{AB}$ operation is distance 2 from a vertex with status D, and hence receives status A or B. In particular, since every node in $S'_A \cup S'_B$ has an attacher, it follows that $S'_A \cup S'_B \subseteq S^*_A \cup S^*_B$.

Further, consider a vertex $f \in V(T')$ that has two leaf-neighbors with status B in S', and show that both these neighbors must have status B in S^*. (Both were attachers for $\mathcal{T}_{AB}$; if one has status A then f has status C in S^* by Lemma 3, but then there is a problem with the other.) Thus f has status A or D. It follows that $S'_A \cup S'_D \subseteq S^*_A \cup S^*_D$.

The above two inclusions imply that $S'_A \subseteq S^*_A$ and $S'_C \supseteq S^*_C \cap V(T')$. Since $|S'_A| = |S'_C|$ and $|S^*_A| = |S^*_C|$, it follows that there is equality in these two inclusions. Hence, by the above inclusions, S^* and S' agree on $V(T')$. $\square$

Now, fix an operation $\mathcal{T}_j$ and attacher status L, and define a tree T as follows. Start with the path P_2 labeled so as to have a vertex l of status L. Then let (T, S) be the ρ-i-tree obtained by applying $\mathcal{T}_j$ to l four times. Let M denote the four new neighbors of l.

Consider any construction of (T^*, S^*). Since on creation a vertex has degree at most 3, and l has degree at least 4 in T, there is a vertex $m \in M$ that is created after l. Let T_m (resp. T^*_m) denote the subtree of T (resp. T^*) with vertex set m and all vertices separated from l by m.

Note that whenever a vertex of status B is created, it is the one attached to an existing vertex. So we may assume that the operations that create the vertices of status B in $V(T^*_m) \setminus V(T_m)$ all occur after all vertices of T_m exist. But the only way to create T_m is to use $\mathcal{T}_j$ applied to l. That is, the operation is essential.

On the other hand, even though the initial P_2 is needed to produce all labelings (such as the P_2 with labels A and C), one can do without it in producing all (ρ, i)-trees:

Observation 11 *If T is a (ρ, i)-tree, then for some ρ-i-labeling S there is a construction of (T, S) starting with P_1.*

Proof. The proof is by induction on the order of T. If $T = P_1$ the result is trivial. So suppose that T is a (ρ, i)-tree, and assume the result holds for all smaller trees. For some ρ-i-labeling S there is a construction of (T, S) (by Theorem 7). Suppose the construction starts with the path x, y, where x has status A and y has status C. If x is a leaf, then we can start with x (of status D), attach y using $\mathcal{T}_1$, and continue as before, since anything that can be attached to a vertex of status C can be attached to a vertex of status B. Similarly, if only operation $\mathcal{T}_1$ is applied to x, we can start with x. Therefore we may assume $\mathcal{T}_2$ or $\mathcal{T}_5$ is applied to x.

If $\mathcal{T}_2$ is used to attach a path u, v to x, then v has status D and u has status B, so we can start with v, attach u, x, y using $\mathcal{T}_1$ and $\mathcal{T}_3$, and continue as before. Suppose $\mathcal{T}_5$ is used to attach a path u, v, w to x: If we root T at y and let $T' = T_v$, then (T', S') is a ρ-i-tree ($|S'_A| = |S'_C|$ since vertices of status A and C occur in adjacent pairs). By the inductive hypothesis there is a construction of (T', S^*) for some S^*, starting with P_1. We can extend this construction by attaching u, x, y to v as follows: by using $\mathcal{T}_4$ if v has status B or C in S^*, or by using $\mathcal{T}_1$ and $\mathcal{T}_3$ otherwise. Finally, we construct the rest of T as before. □

3.2 Strong equality

It can be shown that the graphs with strong equality—which were first characterized in [10] (where they were denoted by $\mathcal{T}_2$)—are those that can be attained by using only the three operations: $\mathcal{T}_1$ with attacher A, $\mathcal{T}_3$, and $\mathcal{T}_4$.

4 Building ρ-γ_t-Trees

We consider here ρ-γ_t-trees. Recall that the smallest (γ, γ_t)-tree is P_4. It has a unique labeling as a ρ-γ_t-tree: leaves with status C and internal vertices with status A. Now, define three operations.

- **Operation $\mathcal{U}_1$.** Take a vertex y of status B which has no neighbor of status C, add a labeled P_4, and join y to a leaf of the P_4.
- **Operation $\mathcal{U}_2$.** Add a labeled P_4, and join a vertex y of status B to an internal vertex of the P_4.
- **Operation $\mathcal{U}_3$.** Attach to a vertex y of status B or C a vertex of status B and join that vertex to an internal vertex of a labeled P_4.

These operations are illustrated in Figure 4.

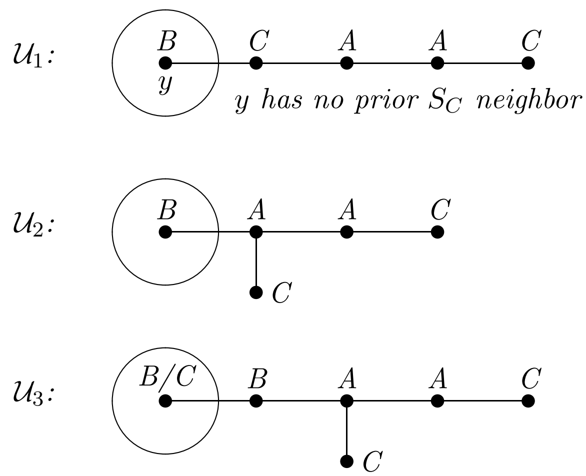

Figure 4: The three $\mathcal{U}_i$ operations

Theorem 12 *A labeled tree is a ρ-γ_t-tree if and only if it can be obtained from a labeled P_4 using the operations $\mathcal{G}_1$, $\mathcal{G}_4$, $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$.*

Proof. It is clear that these operations preserve a (ρ, γ_t)-labeling. So we need to show that any ρ-γ_t-tree can be constructed. The proof is by induction on the order of the tree (with the base case of order 4 trivial). We need to identify a set P of vertices that can be pruned to leave a ρ-γ_t-tree, and an operation $\mathcal{R}$ that restores the pruned vertices.

By Lemma 3, there is no vertex of status D. Thus S_A is a total dominating set and S_C is a packing. By the same lemma there is a matching between S_A and S_C. It follows that every leaf has status B or C and every vertex adjacent to a leaf has status A. If there is a leaf in S_B, then P being that vertex and $\mathcal{R} = \mathcal{G}_1$ works for the induction. So assume that every leaf is in S_C.

Let $Q = u, v, w, x, \ldots$ be a diametrical path. Then $u \in S_C$ and $v \in S_A$. Since the leaves form a packing, v has degree 2 and w is in S_A. If w has another neighbor in S_A, then P being $\{u, v\}$ and $\mathcal{R} = \mathcal{G}_4$ works. So assume that w has no other neighbor in S_A.

Suppose that x is in S_C. Then x's other neighbors are in S_B, and indeed, by the maximality of Q, both x and w have degree 2. Thus P being $\{u, v, w, x\}$ and $\mathcal{R} = \mathcal{U}_1$ works.

So suppose that x is in S_B. Let u' be the neighbor of w that is in S_C: by the maximality of Q, u' is a leaf and w has degree 3. If x has another neighbor in A, then P being $\{u, v, w, u'\}$ and $\mathcal{R} = \mathcal{U}_2$ works. But if x has no other neighbor in A then, by the maximality of Q, it has degree 2, and so P being $\{u, v, w, x, u'\}$ and $\mathcal{R} = \mathcal{U}_3$ works. □

5 Other Constructions

There are many possible variations of the idea. One can, for instance, characterize the class of trees T for which $\gamma(T) = \gamma_t(T) = i(T)$, by using six labels A, B, C, A', B', C' and letting $|S_A \cup S_{A'}| = |S_C \cup S_{C'}|$, $S_A \cup S_{A'}$ be a total dominating set, $S_C \cup S_{C'}$ a packing, $|S_C| = |S_{A'} \cup S_{B'}|$, and $S_{A'} \cup S_{B'} \cup S_{C'}$ an independent dominating set. We omit the details.

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