An Upper Bound for the Ramsey Numbers $r(K_3, G)$

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Abstract

The Ramsey number $r(H, G)$ is defined as the minimum N such that for any coloring of the edges of the N-vertex complete graph K_N in red and blue, it must contain either a red H or a blue G. In this paper we show that for any graph G without isolated vertices, $r(K_3, G) \leq 2q + 1$ where G has q edges. In other words, any graph on $2q + 1$ vertices with independence number at most 2 contains every (isolate-free) graph on q edges. This establishes a 1980 conjecture of Harary. The result is best possible as a function of q.

1 Introduction

For graphs G and H, the Ramsey number $r(H, G)$ is defined as the minimum number N such that for any coloring of the edges of the N-vertex complete graph K_N in red and blue, it must contain either a red H or a blue G. Harary conjectured that $r(K_3, G) \leq 2q + 1$, where q is the number of edges of G. This inequality is best possible, since Chvátal [1] showed that $r(K_3, T_{n+1}) = 2n + 1$ for any tree T_{n+1} on n edges. Also, it is well-known that $r(K_3, K_p) < 2\binom{p}{2} + 1$.

Erdős, Faudree, Rousseau and Schelp [2] showed that $r(K_3, G) \leq \lceil 8q/3 \rceil$. Sidorenko [3] improved this by showing that $r(K_3, G) \leq 5q/2 - 1$ (for $q \geq 4$). In this paper we establish Harary's conjecture.

Theorem. *For any graph G with q edges and without isolated vertices, $r(K_3, G) \leq 2q + 1$.*

In other words, any graph on $2q + 1$ vertices with independence number at most 2 contains every (isolate-free) graph on q edges.

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2 Preliminaries

Let G have q edges, p vertices and minimum degree δ. We prove the result by induction on q. In particular, let R be such that a red-blue coloring of K_R without a red K_3 always contains a blue copy of every graph on fewer edges than G and yet doesn't necessarily contain G. Then we find an upper bound on R, assuming it exists.

Like Sidorenko [3], we focus on the minimum degree. He established that $R \leq 2q$ when $\delta = 1$, so we will assume that $\delta \geq 2$.

Further, we use the same two-case approach as Sidorenko. Call a vertex a δ-vertex if it has degree δ. Then the first case is when G has adjacent δ-vertices.

Lemma 1. *If G has two adjacent δ-vertices then $R \leq 2q$.*

Proof: Let u_1 and u_2 be adjacent δ-vertices with neighborhoods W_1 and W_2 (themselves excluded). Let G' be the resultant graph when one contracts u_1u_2 to form w. Consider a coloring of K_R that includes a blue G' (but no red K_3), and let X denote the remaining vertices.

Suppose there exist distinct vertices $x_1, x_2 \in X$ with x_i blue-adjacent to all of W_i $(i = 1, 2)$. Consider the three vertices x_1, x_2 and w. It is easy to see that if any two of these are joined by a blue edge, then we obtain a blue G. Therefore these three vertices form a red K_3, a contradiction.

Thus there exists an $i \in \{1, 2\}$ such that every vertex in X, except perhaps one, is red-adjacent to some vertex in W_i. We claim that a vertex has red-degree at most $p - 1$; for otherwise its red-neighborhood would contain a blue K_p and hence G. Thus $|X| \leq (p-1)|W_i| + 1$. Hence

$$R \leq (p-1) + (p-1)(\delta-1) + 1 = \delta p - (\delta - 1) \leq 2q - (\delta - 1),$$

as required. $\square$

Now, consider a coloring of K_R without a red K_3 and without a blue G. Let t denote the size of the largest blue clique. It is trivial that the maximum red-degree is at most t, and that $t \leq p - 1$. Another simple bound is:

Lemma 2. $R \leq p + \delta t - 1$.

Proof: Let v be any δ-vertex. Then in K_R there is a blue $G - v$, with the remaining vertices constituting X, say. Let $w_1, \ldots, w_\delta$ be v's neighbors in this copy of $G - v$. If this copy does not directly extend to a blue G, then every vertex in X is red-adjacent to one of the w_i. Thus the red neighborhoods of the w_i cover X, and hence $|X| \leq \delta t$. $\square$

3 Independent δ-vertices

From now on we assume that the δ-vertices form an independent set of size s. We focus on the largest blue clique T in the coloring of K_{2q+1}, and argue that this can be extended to a blue copy of G. In this copy, the non-δ-vertices lie in T, while some δ-vertices lie in T and some outside. We use a greedy approach to show that there must be enough good vertices outside T.

We assume that the coloring of K_{2q+1} does not contain a red K_3. Let $Y = V(K_{2q+1}) - T$ have cardinality y, and let $f = p - t$ denote the number of vertices to be placed outside T. The proof is in three parts. We first establish Conditions which ensure that T can be extended to a copy of G. We then derive some useful bounds, and verify that $y = 2q + 1 - t$ satisfies the Conditions for $\delta \geq 3$. Finally, we handle the case when $\delta = 2$.

3.1 Conditions for Extension

Suppose $y \geq t$. For I a δ-subset of T, let g_I denote the number of vertices in Y which are blue-adjacent to all of I. Every vertex in T is blue-adjacent to at least $y - t$ vertices in Y. We will assume that we have equality here. (For example, we may forbid our blue copy of G to use certain edges.) Thus:

$$y - \delta t \leq g_I \leq y - t.$$

Further, let $\bar{g}$ denote the average value of g_I.

Now, assume $t \geq p - s$. Consider a possible placement in T of the non-δ-vertices of G. Let $I_1, I_2, \ldots, I_s$ denote the resulting sets to which we need to attach δ-vertices $w_1, w_2, \ldots, w_s$. Assume $g_{I_1} \geq g_{I_2} \geq \ldots \geq g_{I_s}$. We can place $w_{f+1}, w_{f+2}, \ldots, w_s$ inside T without problems. Then we place w_f outside T; this requires $g_{I_f} \geq 1$. Next we place w_{f-1} outside T; if $g_{I_{f-1}} \geq 2$ then such a vertex is guaranteed to exist. So a greedy algorithm completes the placement of the δ-vertices provided:

$$g_{I_j} \geq f - j + 1 \qquad \text{for } 1 \leq j \leq f. \tag{1}$$

For this it is sufficient that

$$\sum_{i=1}^{s} g_{I_i} \geq s(f - j) + (j - 1)(y - t - f + j) + 1 \quad \text{for } j = 1, 2, \ldots f. \tag{2}$$

(See Figure 1.) The right-hand side of this expression is maximized at either $j = 1$ or $j = f$ where it has values $s(f-1)+1$ and $(f-1)(y-t)+1$. Further, by the above lower bound on g_I, if $y - \delta t \geq 0$ then we need only worry about $j \leq f - (y - \delta t)$.

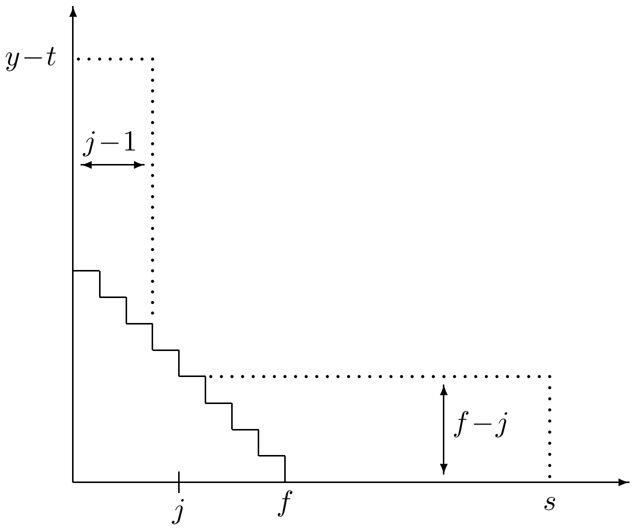

Figure 1: We need the Staircase to be under curve $\{g_{I_j}\}$

By standard reasoning there exists a placement such that $\sum_{j=1}^{s} g_{I_j} \geq s\bar{g}$. Hence:

Lemma 3. *Assume $y \geq t \geq p - s$. Then the following two conditions guarantee that the $\{g_{I_j}\}$ satisfy Condition (1), and thus that T can be extended to a copy of G:*

C1: $f \leq \bar{g}$, *and*
C2: $f(y - t) \leq s\bar{g}$.

If $\sigma = y - \delta t \geq 0$ then we may replace **C2** *by*

C2′: $\sigma s + (f - \sigma)(y - t - \sigma) \leq s\bar{g}$.

3.2 Verification of Conditions

Recall that $0 < s, t < p$ and $f = p - t$. Observe that $2q \geq (\delta + 1)p - s$. By the independence of the δ-vertices, $q \geq \delta s$. Further we may assume that $2q < p + \delta t$, else we are done by Lemma 2. Thus:

$$p + \delta t > 2q = y + t - 1 \geq \max(2\delta s,\ (\delta + 1)p - s). \tag{3}$$

In particular:

Lemma 4.
a) $s \geq \delta f$;
b) $y \geq p2\delta(\delta + 1)/(2\delta + 1) - t + 1$;
c) $t > f(2\delta - 2)$.

Proof: Part (a) follows from $p + \delta t \geq (\delta + 1)p - s$. The lower bound for y is minimized when $2\delta s = (\delta + 1)p - s$; this yields (b). Now $p + \delta t \geq y + t \geq p2\delta(\delta + 1)/(2\delta + 1)$, so that $p/t \leq (2\delta^2 + \delta)/(2\delta^2 - 1)$. This implies that $p/t < (2\delta - 1)/(2\delta - 2)$ which rearranged gives (c). □

Hence $y \geq t \geq p - s$.

Lemma 5. *If $\bar{g} \geq (y - t)/\delta$, then Conditions* **C1** *and* **C2** *are satisfied.*

Proof: Condition **C1** holds since $\bar{g} \geq (y - t)/\delta \geq (2q - 2t)/\delta \geq p - 2t/\delta \geq f$. Condition **C2** holds since $s \geq \delta f$ (by above lemma). □

So we need a bound on $\bar{g}$. Let d_i denote the blue-degree into T of the i^{th} vertex of Y. Then $\sum_I g_I = \sum_i \binom{d_i}{\delta}$, while $\sum_i d_i = t(y - t)$. Hence:

$$\bar{g} \geq y \binom{t(y-t)/y}{\delta} \Big/ \binom{t}{\delta} = (y - t) \prod_{j=1}^{\delta-1} \left(1 - \frac{t^2}{y(t-j)}\right). \tag{4}$$

The above bound for $\bar{g}/(y - t)$ is minimized at y as small as possible; so take $y = (\delta - 1)t$, a lower bound by Lemma 4. Then it is minimized for t as small as possible; so take $t = 2(\delta - 1)$, a lower bound by Lemma 4. Thus $\bar{g}/(y - t) \geq \prod_{j=1}^{\delta-1}(1 - 2/(2\delta - 2 - j)) = (\delta - 3)/(4\delta - 6)$.

For $\delta \geq 6$ we are thus home. If we are more careful, we can show that $\bar{g} \geq (y - t)/\delta$ for $\delta \geq 3$ (with one exceptional case). When $\delta = 2$ we must go back and verify the conditions of Lemma 3 directly. The details are given below.

3.3 Arithmetical Details

From Lemma 4 and Bound 4 we obtain that $\bar{g} \geq (y - t)/\delta$ for $3 \leq \delta \leq 5$ except when $(\delta, f) = (3, 1)$ as follows. If $\delta = 5$, then $y \geq 60p/11 - t \geq 49(t + 1)/11$, and $t \geq 9$. The expression $t^2/((t+1)(t-j))$ is minimized at t as small as possible. So plug in lower bounds for y and t and get $\bar{g}/(y - t) \geq 0.253$. Similarly for $\delta = 4$: $y \geq 40p/9 - t \geq 31(t + 1)/9$ and $t \geq 7$, and plug in to get $\bar{g}/(y - t) \geq 0.251$. If $\delta = 3$ and $f \geq 2$, then $t \geq 9$. Since $y \geq 24(t+f)/7 - t \geq 17t/7 + 48/7$, it holds that $y(t - j) \geq y(t - 2) \geq (17t^2 + 14t - 96)/7 \geq 17t^2/7$. Hence $\bar{g}/(y - t) \geq (10/17)^2 \geq 0.346$. When $(\delta, f) = (3, 1)$ we merely need $\bar{g} > 0$ (by Equation 2). For this it is sufficient that $t(y - t) > 3y$. The expression $E = t(y - t) - 3y$ is minimized at y as small as possible, say $y = 17t/7 + 24/7$; and then at t as small as possible, viz. $t = 5$. E's value there is 43/7.

Thus it remains to verify the conditions when $\delta = 2$. Note that $p \leq 10t/7$ (cf. proof of Lemma 4).

We consider first the case when $f = 1$. Here we need $\bar{g} > 0$ (by Equation 2). For this it is sufficient that $t(y - t) > 2y$. The expression $E = t(y - t) - 2y$ is minimized at y as small as possible, say $y = 12(t + 1)/5 - t + 1 = 7t/5 + 17/5$ (Lemma 4); and then at t as small as possible. If $t \geq 4$ then $E \geq 2$. The case when $t = 3$ is easily dispensed with. (Recall that $r(K_3, K_4) = 9$.) So from now on we assume that $f \geq 2$, and thus $t \geq 5$ (by Lemma 4).

We next verify Condition **C1**. By the bound of (4), it suffices to show that $f \leq (y - t)(yt - t^2 - y)/(yt - y)$. By rearranging it suffices to show that

$$y^2t - yt^2 + t^3 - ytp \geq y^2 - yp.$$

Since $y \leq p + t$ the right-hand side of this expression is at most $t(p + t)$. On the other hand, the left-hand side L is minimized at the smallest value of y ($\partial L/\partial y = t(2y - p - t)$ and $y \geq (p + t)/2$ by Lemma 4). So take $y = 12p/5 - t$ (a lower bound by Lemma 4), where $L = t(84p^2 - 155pt + 75t^2)/25 \geq t(79p^2 - 150pt + 75t^2)/25 = t(4p^2 + 75f^2)/25$. So it is sufficient that $4p^2 + 300 \geq 25(p + t)$, which is true.

Finally we verify Condition **C2**. Let $s = \alpha f$. By Lemma 4, $\alpha \geq 2$. We need to establish that $yt - y \leq \alpha(yt - t^2 - y)$, or equivalently that $F = t(\alpha(y - t) - y) - (\alpha - 1)y \geq 0$. The expression F is minimized at y as small as possible ($\partial F/\partial y = (t - 1)(\alpha - 1)$). We start with the case $y \leq 2t$. Then $\alpha \geq 3$ since $s \geq 3p - y - t$ by Inequality (3). As $y \geq 3p - \alpha f - t$,

$$\alpha(y - t) - y \geq 3p\alpha - \alpha^2 f - \alpha t - \alpha t - 3p + \alpha f + t = (\alpha - 3)(p - s) + t \geq t.$$

Thus it remains to verify that $t^2 \geq (\alpha - 1)y$. Since $y \leq 2t$, for this it is sufficient that $t \geq 2s/f$. By Inequality (3), $y \geq 4s - t$ so that $t \geq 4s/3$. As $f \geq 2$ we are done.

Next we consider the case $y \geq 2t$. Then for $F \geq 0$ it is sufficient that $\alpha \geq 2(t - 1)/(t - 2)$. Hence if $\alpha > 5/2$ and $t \geq 6$ then we are done. The case $t = 5$ and $\alpha > 5/2$ is easily handled. (Since it follows that $p = 7$ and $f = 2$ whence $\alpha \geq 3 > 8/3$, as required.)

So consider $\alpha \leq 5/2$ and Condition **C2′**. By plugging in the bound of (3) and multiplying through by $y(t - 1)/t$, it is sufficient to show that

$$st^3 - sty - pyt^2 + y^2t^2 - yt^3 + pyt - y^2t + yt^2 \geq 0.$$

The left-hand side L is minimized at y as small as possible ($\partial L/\partial y = t((2y - p - t)(t - 1) - s)$). By Inequality (3), $y \geq 3p - s - t \geq 3p - 5(p - t)/2 - t$. Using this bound it then follows that L is minimized at s as small as possible, so take $s = 2(p - t)$. Simplifying, the condition reduces to verifying that $5t^2 - 9t - pt - 3p \geq 0$. This is valid since $p \leq 10t/7$ and $t \geq 5$.

Noted added in Proof

The result in this paper was obtained earlier and independently by A.F. Sidorenko by different means.

References

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