The Fully Weighted Toughness of a Graph

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Abstract

The toughness of a graph $G$ is defined to be the minimum value of $\frac{|S|}{k(G - S)}$, where $k(G - S)$ denotes the number of components of $G - S$ and the minimum is taken over all cut-sets $S \subseteq V(G)$. In this paper we propose a version for weighted graphs that depends on the weights in both $S$ and $G - S$. Apart from considering bounds and basic properties, our focus is on the problem of assigning weights so as to maximize the parameter.

1 Introduction

50 years ago Chvátal [2] defined the toughness $\tau(G)$ of a graph $G$ to be the minimum value of $\frac{|S|}{k(G - S)}$, where $k(G - S)$ denotes the number of components of $G - S$ and the minimum is taken over all cut-sets $S \subseteq V(G)$ (where a cut-set is a set such that $G - S$ is disconnected). Over the years, toughness has been investigated by many authors, with the focus somewhat on the relationship with the existence of cycles and other subgraphs. Most famous is the unresolved conjecture from [2] that there is a toughness threshold for a graph to be Hamiltonian. But there are also calculations and bounds; for example, Matthews and Sumner [8] showed that the toughness is exactly half the connectivity if the graph is claw-free. For more information, see the survey [1], for example.

Recently, both Katona, Kovács, and Varga [7] and Shi and Wei [12] defined the weighted toughness. They introduced a nonnegative weighting $w$ on the vertex set, and replaced $|S|$ by $w(S)$ where $w(S)$ denotes the sum of the weights of the vertices in $S$. In this paper we propose an alternative way to generalize toughness to weighted graphs. In the spirit of measures of vulnerability such as, for example, weighted integrity [10], we propose considering not just the weights of the vertices removed but also the weights of the vertices that remain. Our motivation is that, if one has for example the graph $2K_2$ where two of the vertices have weight 100 and two have weight 1, it seems reasonable to view the situation with the two
vertices of weight 100 in separate components as more “disconnected” than the situation with the two vertices of weight 100 in the same component.

Thus we introduce what we call the fully weighted toughness of a graph. We proceed as follows. In Section 2 we provide the definitions and basic properties of the parameter. In Section 3 we investigate the problem of assigning weights to the vertices of an unweighted graph, especially a tree, so as to maximize the fully weighted toughness. Finally, in Section 4 we provide some brief thoughts on future work.

2 The Basics of Fully Weighted Toughness

2.1 The Definitions

We consider graphs with a weighting on the vertices. Throughout, we require that each individual weight be nonnegative and that the total weight is positive.

As noted above, the original definition of toughness considers the ratio for cut-sets $S$ of $|S|/k(G - S)$. In this case it is immediate that one can replace the numerator by the sum of the weights of the vertices of $S$, which yields the weighted toughness of [7, 12]. But what about the denominator?

For a graph $H$ where the vertices have weighting $w$, we define:

$$k_w(H) \text{ is the sum, over the components of } H, \text{ of the maximum weight of a vertex in each component.}$$

Note that if all the weights are 1, then this definition reverts to the original definition of $k(H)$.

Further, the original definition of toughness considered only $S$ that are cut-sets, and so the toughness of the complete graph is considered to be infinite. As suggested in [3] and elsewhere, it is also reasonable to emulate the definition of connectivity, where the complete graph $K_n$ has connectivity $n - 1$. That is, one may also allow sets $S$ such that $G - S$ is a single vertex. We adopt that approach here. Recall that $w(S)$ denotes the sum of the weights of the vertices in set $S$. Then we define:
Definition 1  The fully weighted toughness of a graph $G$ with weighting $w$ is defined to be

\[ \tau_w(G) = \min \left\{ \frac{w(S)}{k_w(G - S)} \right\}, \]

where the minimum is taken over all $S \subseteq V(G)$ such that $S$ is a cut-set of $G$ or $G - S$ has one vertex, and such that $k_w(G - S) > 0$. A set achieving the minimum is called a tough set.

Note that due to the form of the tough ratio, one can scale all weights by the same positive factor and the fully weighted toughness does not change. Further, one can restrict the sets $S$ that need to be considered. As defined for example in [11], a cut-set is strong if $G - S'$ has fewer components than $G - S$ for all $S'$ strictly contained in $S$.

Lemma 1  For any graph $G$, there exists a tough set $S$ such that either $S$ is a strong cut-set or $G - S$ is a single vertex and that vertex is a dominating vertex.

Proof.  If cut-set $S$ has a subset $S'$ where $G - S'$ has the same number of components as $G - S$, then $w(S') \leq w(S)$ and $k_w(G - S') \geq k_w(G - S)$. If $G - S$ is a single vertex $v$ and there exists a vertex $x$ not adjacent to $v$, then $S' = S - \{x\}$ is a cut-set with $w(S') \leq w(S)$ and $k_w(G - S') \geq k_w(G - S)$. In each case, the ratio for $S$ would be at least the ratio for $S'$. QED

It is to be noted that allowing a set $S$ that leaves a single vertex affects the fully weighted toughness of more than just the complete graph. Consider for example the path $P_3$ on 3 vertices. If only cut-sets are allowed, one can place any positive weight on the middle vertex and zero on the end-vertices and get infinite fully weighted toughness. But if one allows the set $S$ to be the two end-vertices, then it is straight-forward to show (see Theorem 1) that the fully weighted toughness of $P_3$ is at most 1, which coincides with the fact that the ordinary toughness of a bipartite graph is at most 1.

The fully weighted toughness problem (that is, on input graph $G$, weighting $w$ and threshold $s$, is $\tau_w(G) \leq s$) is of course NP-hard in general, since having all weights 1 reduces to the ordinary toughness.
2.2 Some Bounds and Examples

We start with the calculation of the fully weighted toughness of the complete multipartite graphs.

**Lemma 2** Let $G$ be a complete $r$-partite graph with partite sets $X_1, \ldots, X_r$. If $w$ is a weighting with total weight $W$, then

$$\tau_w(G) = \frac{W}{\max_{1 \leq i \leq r} w(X_i)} - 1.$$ 

**Proof.** By Lemma 1 the only sets $S$ that need to be considered are each the union of all but one partite set. If $S = V(G) - X_i$, then $w(S) = W - w(X_i)$ while $k_w(S) = w(X_i)$; thus $\tau_w(G) \leq W/w(X_i) - 1$. This ratio is minimized when $w(X_i)$ is maximized. QED

We use averaging for several bounds. In particular we need the elementary property of the mediant.

**Lemma 3** If $a_1, \ldots, a_k$ are nonnegative reals and $b_1, \ldots, b_k$ positive reals, then the mediant $M = (\sum a_i)/(\sum b_i)$ of the fractions $a_i/b_i$ is at least the minimum value of $a_i/b_i$ and at most the maximum value of $a_i/b_i$. Furthermore, if the minimum and maximum fractions are not equal, then $M$ lies strictly between the two.

Our first bound involves the chromatic number.

**Theorem 1** If graph $G$ has chromatic number $r$, then $\tau_w(G) \leq r - 1$ for any weighting $w$.

**Proof.** Consider a weighting $w$ of $G$ with total weight $W$. Let $A_1, \ldots, A_r$ be the color classes of a (proper) coloring of $G$. Consider removing the set $V(G) - A_i$ from the weighted graph $G$. The resultant graph has no edges, and hence $\tau_w(G) \leq (W - w(A_i))/w(A_i)$. Taking the mediant of these $r$ ratios, we have

$$\tau_w(G) \leq \frac{\sum_{i=1}^r W - w(A_i)}{\sum_{i=1}^r w(A_i)} = \frac{rW - W}{W} = r - 1.$$
This gives the desired bound. QED

As noted in Lemma 2, the bound of Theorem 1 is achievable by any complete \( r \)-partite graph if one puts equal weight on each partite set. For example, the complete graph minus an edge, \( K_n - e \), can have fully weighted toughness \( n - 2 \) by placing \( \frac{1}{2} \) on both ends of \( e \) and 1 on every other vertex. In particular, this example shows that the fully weighted toughness of a graph is not bounded above by \( n/\alpha - 1 \) (where \( n \) denotes the order and \( \alpha \) the independence number), as it is for ordinary toughness.

For a tree \( T \), Pippert [9] observed that the ordinary toughness is determined by the maximum degree, namely \( \tau(T) = 1/\Delta(T) \). The same is not true for fully weighted toughness, but we observe next that one may at least assume that the set \( S \) is a singleton.

We will need the following concepts. Given a tree \( T \) and a set \( S \) of vertices, we define the \( S \)-portion of \( T \) to be the subgraph induced by the vertices of \( S \) and all vertices on paths between vertices of \( S \). This is a subtree of \( T \). Further, we define a vertex \( x \) of \( S \) to be \( S \)-extremal, or simply extremal, if it is an end-vertex in the \( S \)-portion; in other word, there exists a bridge that separates \( x \) from \( S - \{x\} \).

**Theorem 2** The fully weighted toughness of a tree \( T \) that is not a star is achieved by removing a single vertex.

**Proof.** Consider a tough set \( S \) of minimum cardinality. By Lemma 1 we may assume it is a strong cut-set; thus \( S \) does not contain any end-vertex. Suppose \( S \) is not a singleton.

Consider an \( S \)-extremal vertex \( x \). Let \( A \) denote the set of vertices \( v \) of \( V(T) - S \) such that in \( T \) every path from \( v \) to a vertex of \( S - \{x\} \) goes through \( x \). Let \( B \) denote the set of vertices \( v \) of \( V(T) - S \) such that in \( T \) every path from \( v \) to \( x \) goes through at least one vertex of \( S - \{x\} \). And let \( C \) be the remainder of \( V(T) - S \). Note that \( A \) and \( B \) are nonempty. On the other hand, \( C \) might be empty, but if nonempty then the subgraph induced by \( C \) is connected. For a set \( X \) of vertices, let \( m_w(X) \) denote \( k_w(T[X]) \) where \( T[X] \) is the subgraph of \( T \) induced by \( X \). Let \( \alpha = m_w(A) \), \( \beta = m_w(B) \), and \( \gamma = m_w(C) \), where we set \( \gamma = 0 \) if \( C \) is empty. Figure 1 gives a depiction.
Figure 1: Splitting a tree

Now, if one removes vertex $x$ from $T$, then the components of $T - x$ are those induced by $A$ and one component consisting of the vertices of $B$, $C$, and $S - \{x\}$. The maximum weight of a vertex in the latter component is at least $\gamma$. It follows that $m_w(T - x) \geq \alpha + \gamma$. By a similar reasoning it follows that $m_w(T - (S - \{x\})) \geq \beta + \gamma$. On the other hand, $m_w(T - S) = \alpha + \beta + \gamma$.

Thus the fully weighted toughness of the tree $T$ is bounded above by both $\tau_x = \frac{w(x)}{k_w(G - x)}$ and $\tau_{S-x} = \frac{w(S - x)}{k_w(G - (S - x))}$ where

$$\tau_x \leq \frac{w(x)}{\alpha + \gamma} \quad \text{and} \quad \tau_{S-x} \leq \frac{w(S - \{x\})}{\beta + \gamma}.$$

By considering the mediant of $\tau_x$ and $\tau_{S-x}$, it follows that the fully weighted toughness of the tree is at most

$$\tau_w(T) \leq \frac{w(x) + w(S - x)}{\alpha + \gamma + (\beta + \gamma)} = \frac{w(S)}{\alpha + \beta + 2\gamma} \leq \frac{w(S)}{\alpha + \beta + \gamma} = \tau_w(T).$$

By Lemma 3 both $\tau_x$ and $\tau_{S-x}$ equal $\tau_w(T)$, and hence both $\{x\}$ and $S - \{x\}$ contradict the choice of $S$. QED

In particular, the above theorem shows that the fully weighted toughness of a tree can be computed in polynomial time.

Given an unweighted graph, one can ask for the weights that minimize or maximize the fully weighted toughness. The former question is trivial. One can simply put weight 0 on a cut-set (and weight other vertices arbitrarily), and then the fully weighted toughness is 0. But the latter question is more interesting, and that is what we consider next.
3 The Maximum Fully Weighted Toughness of a Graph

Consider maximizing the fully weighted toughness of an unweighted graph. For a graph $G$, we define $\text{MFWT}(G)$ as the maximum value of $\tau_w(G)$ over all weightings $w$ of $G$. Since one can give every vertex weight 1, it follows that $\text{MFWT}(G) \geq \tau(G)$ assuming $G$ is not complete. At the same time, if vertex $v$ has maximum weight, then one can choose $S = V(G) - \{v\}$, and so $\text{MFWT}(G) \leq (n-1) w(v)/w(v) = n - 1$, where $n$ is the order of $G$.

It is immediate from Lemma 2 that if $G$ is a complete $r$-partite graph then $\text{MFWT}(G) = r - 1$, attained by making the weight on each partite set equal. There are also graphs where the maximum fully weighted toughness is arbitrarily small. Consider the octopus $O_s$ defined by starting with the star with $s$ leaves and subdividing each edge exactly once.

**Lemma 4** For $s \geq 1$, $\text{MFWT}(O_s) = 1/\sqrt{s}$.

**Proof.** Since $O_1$ is a complete bipartite graph, by the above comment $\text{MFWT}(O_1) = 1$. So consider the case $s \geq 2$. Since $O_s$ is not a star, it follows from Theorem 2 that the only $S$ one need consider are the singletons $\{v\}$ where $v$ is not an end-vertex. Thus the weights of the end-vertices appear only in the denominator of ratios, and since we are interested in the maximum value of the ratios, these weights may be assumed 0.

Let $w_1, \ldots, w_s$ denote the weights of the degree-2 vertices and $y$ the weight of the center. Then we have $\tau_w(O_s) \leq w_i/y$ for each $i$ (since the maximum component weight is at least $y$), and $\tau_w(O_s) \leq y/\sum w_i$. Then it can easily be checked that the biggest value of the minimum of these ratios is achieved where all the $w_i$ are equal, and $y = \sqrt{s}w_1$. That is, $\text{MFWT}(O_s) \leq 1/\sqrt{s}$. But the stated weights also show equality. QED

Figure 2 shows an optimal weighting of $O_5$.

We continue with an example that further demonstrates the contrast with ordinary toughness. We will need the following lemma.
Figure 2: The octopus $O_5$

**Lemma 5** If vertices $u$ and $v$ of graph $G$ are adjacent and have the same neighbors, then in computing $\text{MFWT}(G)$ one may assume the two vertices get the same weight.

**Proof.** Suppose $w$ is a weighting with $w(u) > w(v)$. Let weighting $w'$ be obtained from $w$ by increasing the weight of $v$ to equal that of $u$. We claim that $\tau_{w'}(G) \geq \tau_w(G)$. Let $S$ be the tough set guaranteed by Lemma 1. Consider first that $S$ is a strong cut-set Then either both $u$ and $v$ are in $S$ or neither is in $S$. If neither is in $S$, then $u$ and $v$ are in the same component of $G - S$, and so $k_w(G - S) = k_{w'}(G - S)$ and $\tau_{w'}(G) = \tau_w(G)$. If both vertices are in $S$, then $w'(S) > w(S)$ and $\tau_{w'}(G) > \tau_w(G)$. Consider second that $G - S$ is a single vertex. Then the smallest ratio for such $S$ is achieved where $G - S$ is a vertex of maximum weight. So the largest tough ratio for such a set is the same for $w'$ as it is for $w$. This proves the claim. QED

**Lemma 6** Let $G$ be the graph formed by the join of clique $K_a$ with disjoint cliques $K_{b_1} \cup \ldots \cup K_{b_s}$ with $s \geq 2$ and $b = \max(b_1, \ldots, b_s)$. Then 

$$
\text{MFWT}(G) = \frac{2ab}{\sqrt{(a-1)^2 + 4ab - (a-1)}}.
$$

**Proof.** Assume $b_1 = b$. By the above lemma, we may assume that for each constituent clique, every vertex in that clique has the same weight. So by scaling, we may assume that each vertex in $K_a$ has weight 1, and each vertex in $K_{b_i}$ has weight $w_i$.  

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By Lemma 1, the fully weighted toughness is achieved either by removing all of $K_a$, or by removing all vertices of $G$ except one vertex of $K_a$. So the fully weighted toughness is given by

$$\min \left\{ \frac{a}{w_1 + \ldots + w_s}, a - 1 + \sum_{i=1}^{s} b_i w_i \right\}.$$  

For a fixed value of $\sum w_i$, the second quantity is maximized at $w_2 = \ldots = w_s = 0$. (That is, we may put all the weight on the biggest clique.) Then the first quantity is decreasing in $w_1$ while the second is increasing in $w_1$. Hence the minimum is maximized where the two quantities are equal. So we have

$$a/w_1 = a - 1 + bw_1.$$ 

One can solve this equation for $w_1$, and hence obtain that the maximum ratio is the stated bound. \textsf{QED}

For example, if $a = 1$ then the value in Lemma 6 simplifies to $\sqrt{b}$, while the ordinary toughness is $1/s$. Figure 3 is an example optimal weighting for the join of $K_1$ with cliques $K_2 \cup K_3$.

![Figure 3: The join of $K_1$ with $K_2 \cup K_3$ and an optimal weighting](image)

Now, if we fix $a$, by calculus the expression in Lemma 6 is increasing in $b$. So for fixed $a$ and fixed order of the graph, the bound is maximized by making $b$ as large as possible; that is, by taking $s = 2$ and $b_2 = 1$ so that $b = n - a - 1$. So, not unexpectedly, the maximum fully weighted toughness of a graph with given connectivity $a$ is achieved by the graph obtained by taking the clique $K_{n-1}$ and adding one new vertex and joining it to $a$ vertices of the clique. Thus we obtain:

**Theorem** 3 If $G$ is a noncomplete graph of order $n$ and connectivity $a$, then

$$\tau_w(G) \leq \frac{2a(n - a - 1)}{\sqrt{(a - 1)^2 + 4a(n - a - 1)} - (a - 1)},$$

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and this bound is sharp.

Since a graph with connectivity 1 is not Hamiltonian, it follows that no value of the fully weighted toughness is sufficient to guarantee Hamiltonicity. But even if one imposes a large value of connectivity, there is no threshold. For, graphs in the above family need not be 1-tough even for given connectivity, but can have arbitrarily large fully weighted toughness.

There are other simple cases where $\text{MFWT}(G) > \tau(G)$. For example:

**Lemma 7** If nontrivial graph $G$ is connected and has a unique tough set for ordinary toughness, then $\text{MFWT}(G) > \tau(G)$.

**Proof.** Assume $S$ is the ordinary tough set. Then for some $\varepsilon > 0$ define weighting $w$ by giving the vertices in $S$ weight $1 + \varepsilon/|S|$ and the other vertices weight 1. Since the total increase in weight is $\varepsilon$, if $X$ is any other candidate set, then $k_w(G - X) \leq k(G - X) + \varepsilon$. Note that if $a, b, c, d$ are positive integers and $a/b < c/d$, then $(a + \varepsilon)/b < c/(d + \varepsilon)$ for all $\varepsilon$ sufficiently small. Thus $(|S| + \varepsilon)/k(G - S) < |X|/(k(G - X) + \varepsilon)$, provided $\varepsilon$ is small enough. That is, $S$ remains the unique tough set, and $\tau_w(G) > \tau(G)$. QED

### 3.1 Graphs That Cannot be Toughened

We show next that for certain graphs their fully weighted toughness is at most their ordinary toughness. One simple example is the 4-cycle. Let $x_1, \ldots, x_4$ be the vertices with weights $w_1, \ldots, w_4$. Then $\{x_1, x_3\}$ is a cut-set where $x_2$ and $x_4$ are in different components, and vice versa. It follows that the fully weighted toughness is always at most the smaller of $(w_1 + w_3)/(w_2 + w_4)$ and $(w_2 + w_4)/(w_1 + w_3)$, which is at most 1, the ordinary toughness.

One can extend the idea to other symmetric graphs. Fix some cut-set $S$ of the graph $G$. Say $G - S$ has $r$ components with vertex sets $S_1, \ldots, S_r$ of orders $x_1, \ldots, x_r$. If $Y$ is a cut-set such that $G - Y$ is isomorphic to $G - S$, then one can number the components $Y_1, \ldots, Y_r$ of $G - Y$ such that $Y_j$ is isomorphic to $S_j$ for $1 \leq j \leq r$. An $S$-family is a collection $Y_1, \ldots, Y_\ell$ of cut-sets such that $G - Y_i$ is isomorphic to $G - S$ for each $i$. We say that an $S$-family rotates if
(i) each vertex of $G$ is in an equal number of the sets $Y_i$ (necessarily $\ell|S|/n$ times)

(ii) for each $j$, the collection $Y_1^j, \ldots, Y_\ell^j$ contains every vertex of $G$ the same number of times (necessarily $\ell x_j/n$ times), where $Y_i^j$ is the component of $G - Y_i$ isomorphic to $S_j$.

We say that the graph is *rotatable* if every cut-set has a rotating family.

For example, it is easy to see that cycles are rotatable. As a specific instance, the 5-cycle has up to symmetry two cut-sets. If $S$ consists of two nonadjacent vertices, and the vertices of the 5-cycle are $a, b, c, d, e$ in order, then here is the $S$-family that rotates:

<table>
<thead>
<tr>
<th>$Y_i$</th>
<th>$Y_1^i$</th>
<th>$Y_2^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, c$</td>
<td>$b$</td>
<td>$d, e$</td>
</tr>
<tr>
<td>$b, d$</td>
<td>$c$</td>
<td>$e, a$</td>
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<tr>
<td>$c, e$</td>
<td>$d$</td>
<td>$a, b$</td>
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<tr>
<td>$d, a$</td>
<td>$e$</td>
<td>$b, c$</td>
</tr>
<tr>
<td>$e, b$</td>
<td>$a$</td>
<td>$c, d$</td>
</tr>
</tbody>
</table>

A rotating family for the three-vertex cut-set is also easily obtained. Another rotatable graph is the balanced complete bipartite graph: if $S$ is for example the cut-set consisting of one partite set, then a rotating $S$-family is given by it and the other partite set. Another example is the rooks graph, defined as the Cartesian product of two complete graphs. (We omit the details.)

**Theorem 4** If noncomplete graph $G$ is rotatable, then $\text{MFWT}(G) = \tau(G)$.

**Proof.** Fix a weighting $w$ of $G$. Consider a tough set $S$ for ordinary toughness. Then by assumption there is a rotating $S$-family $Y_1, \ldots, Y_\ell$. For each $Y_i$, the fully weighted toughness of $G$ is at most $w(Y_i)/k_w(G - Y_i)$. By replacing the maximum in each component by the average in each component, we have that

$$k_w(G - Y_i) \geq \sum_{j=1}^{r} \frac{w(Y_i^j)}{x_j}.$$
Let $M$ be the mediant of the ratios $w(Y_i)/k_w(G-Y_i)$ for $1 \leq i \leq \ell$. As the fully weighted toughness of $G$ is at most each of these ratios, it follows that

$$\tau_w(G) \leq M = \frac{\sum_{i=1}^{\ell} w(Y_i)}{\sum_{i=1}^{\ell} k_w(G-Y_i)} = \frac{\sum_{i=1}^{r} w(Y_i)}{\sum_{i=1}^{r} \sum_{j=1}^{\ell} w(Y_i^j)/x_j}.$$ 

Since each vertex appears in $\ell|S|/n$ of the sets $Y_i$, the numerator simplifies to $\ell|S|W/n$, where $W$ is the total weight on the vertices of $G$. Similarly, the denominator simplifies

$$\sum_{i=1}^{\ell} \sum_{j=1}^{r} \frac{w(Y_i^j)}{x_j} = \sum_{j=1}^{r} \sum_{i=1}^{\ell} \frac{w(Y_i^j)}{x_j} = \sum_{j=1}^{r} \frac{\ell x_j W/n}{x_j} = r \ell W/n.$$ 

Thus

$$\tau_w(G) \leq M \leq \frac{\ell|S|W/n}{r \ell W/n} = \frac{|S|}{r} = \frac{|S|}{k(G-S)} = \tau(G) \leq \tau_w(G),$$

as required. QED

Note that actually the result holds if the graph has one tough set for ordinary toughness which has a rotating family.

### 3.2 Paths

In this section we calculate the maximum fully weighted toughness of the path on at least 3 vertices. The ordinary toughness of such a path is $\frac{1}{2}$. We know that $\text{MFWT}(P_3) = 1$ since it is a complete bipartite graph.

**Theorem 5** For $n \geq 4$, the value $\text{MFWT}(P_n)$ is the (positive) root of the equation $\alpha^2 + \alpha^3 + \ldots + \alpha^{(n-1)/2} = \frac{1}{2}$ if $n$ is odd; and of $\alpha + \alpha^2 + \ldots + \alpha^{(n-2)/2} = 1$ if $n$ is even.

In particular, the maximum fully weighted toughness of such a path lies strictly between $\frac{1}{2}$ and 1. For example, $\text{MFWT}(P_5) = 1/\sqrt{2}$ and $\text{MFWT}(P_6) = (\sqrt{5} - 1)/2$; an optimal weighting of each is given in Figure 4.

**Proof.** Let $w$ be a weighting. Say the vertices of the path are $v_1, \ldots, v_n$ from left to right with weights $w_1, \ldots, w_n$. By Theorem 2 we know that the toughness ratio $\tau_w$ is achieved by removing exactly one of the non-end vertices. Let $\tau_i$
denote the ratio achieved by the removal of vertex $v_i$ for $2 \leq i \leq n - 1$. Our goal is an upper bound for the minimum of these $n - 2$ ratios.

Then $\tau_i = w_i/(x + y)$ where $x$ is the maximum weight to the left of vertex $v_i$ and $y$ is the maximum weight to the right of vertex $v_i$. This quantity is at most $\frac{1}{2}$ if both $x$ and $y$ are at least $w_i$. It follows that the fully weighted toughness of a path is at most $\frac{1}{2}$, unless the weight distribution has no local minimum: that is, the weights are strictly increasing up to the maximum weight, which occurs either once or on two consecutive vertices, and then the weights are strictly decreasing from there on.

Since one can scale all weights and not change the fully weighted toughness, we may assume that the maximum weight is 1. Assume that $w_\ell$ is the leftmost occurrence of the maximum weight and $w_r$ is the rightmost occurrence. As noted above, either $r = \ell$ or $r = \ell + 1$. The weight $w_1$ only appears in the ratio $\tau_2$ and there in the denominator. Thus we may assume $w_1$ is as small as possible, namely 0. Similarly we may assume $w_n = 0$.

**Claim 1** We may assume $\tau_2 = \tau_3 = \ldots = \tau_{\ell-1} = w_2$, and so

$$w_i = w_2^{i-1} + w_2^{i-2} + \ldots + w_2$$

for all $3 \leq i < \ell$.

**Proof.** Assume $\ell > 3$ and consider the weight $w_2$. It appears in only two of the ratios, namely $\tau_2 = w_2/1$ and $\tau_3 = w_3/(1 + w_2)$. Then if $\tau_2 \neq \tau_3$, one can adjust $w_2$ until the ratios are equal; note that this increases the smaller of the two ratios, and thus cannot decrease $\tau_w$. That is, we may assume that $\tau_2 = \tau_3$, and thus that the equation $w_2(w_2 + 1) = w_3$ holds. Note that this adjustment preserves the condition $w_2 < w_3$. 

Figure 4: Optimal weightings of short paths
Similarly, assume \( \ell > 4 \) and consider the weight \( w_3 \). It appears only in the ratio \( \tau_3 \) in the numerator and in the ratio \( \tau_4 \) in the denominator. Then if \( \tau_3 \neq \tau_4 \), we can adjust \( w_3 \) and \( w_2 \) simultaneously, maintaining the equation \( w_2(w_2 + 1) = w_3 \), until \( \tau_3 = \tau_4 \). Again this increases the smaller of the ratios, and does not affect the monotonicity of the weights. Thus we may assume that \( w_4/(w_3 + 1) = w_3/(w_2 + 1) = w_2 \). In particular, the equation \( w_4 = w_2^3 + w_2^2 + w_2 \) holds. The remainder of the claim is proved similarly. \( \triangleright \)

By a similar argument, we may assume \( \tau_{r+1} = \tau_{r+2} = \ldots = \tau_{n-1} = w_{n-1} \), and that

\[
w_{n+1-i} = w_{n-1}^{i-1} + w_{n-1}^{i-2} + \ldots + w_{n-1}
\]

for all \( 3 \leq i \leq n - r \).

### Claim 2

We may assume that

\[
w_2 = w_{n-1}.
\]

**Proof.** By the above discussion \( \tau_w = \min\{w_2, w_{n-1}, \tau_\ell, \tau_r\} \). Note that if \( \ell = r \) then \( \tau_\ell = 1/(w_{\ell-1} + w_{\ell+1}) \); otherwise \( \tau_\ell = 1/(w_{\ell-1} + 1) \) and \( \tau_r = 1/(1 + w_{r+1}) \).

Suppose \( w_2 \neq w_{n-1} \); say \( w_2 > w_{n-1} \). Then one can adjust \( w_2 \) down, which decreases \( w_{\ell-1} \), and this change can only increase \( \tau_\ell \) (while preserving \( \tau_r \) if \( r > \ell \)) and so does not decrease \( \tau_w \). (And note that this cannot affect the monotonicity of the weights.) Thus the claim follows. \( \triangleright \)

Let \( \alpha = w_2 = w_{n-1} \). Let \( A(k) = \sum_{j=1}^k \alpha^j \). Then \( w_i = A(i-1) \) for \( 2 \leq i \leq \ell-1 \) and \( w_i = A(n-i) \) for \( r+1 \leq i \leq n-1 \). Now, assume \( w \) is an optimal weighting; that is, one where \( \tau_w(P_n) \) is maximized. There are two cases.

**Case 1:** This weighting has \( r = \ell \). Out of all such weightings achieving the optimum, choose the one \( w \) where \( \ell \) is as close to the middle as possible. Suppose that \( \ell \neq (n+1)/2 \). Without loss of generality we may assume \( \ell > (n+1)/2 \). Then by the above formulas, we have \( w_{\ell-2} \geq A(n/2 - 2) \) while \( w_{\ell+1} \leq A(n/2 - 2) \). Thus \( w_{\ell-2} \geq w_{\ell+1} \).

We claim that one can swap the weights \( w_{\ell-2} \) and \( w_{\ell+1} \) without decreasing \( \tau_w \). For, this swap affects only the two ratios \( \tau_{\ell-1} \) and \( \tau_\ell \). These change from \( w_{\ell-1}/(1 + w_{\ell-2}) \) and \( 1/(w_{\ell-1} + w_{\ell+1}) \) to \( \tau'_{\ell-1} = 1/(w_{\ell-2} + w_{\ell-1}) \) and \( \tau'_\ell = \)
Then $\tau'_{\ell-1} > \tau_{\ell-1}$ since $w_{\ell-1} < 1$, while $\tau'_\ell \geq \tau_{\ell-1}$ since $w_{\ell-2} \geq w_{\ell+1}$. Thus the new $\tau'_w(G)$ is at least the old $\tau_w(G)$, which contradicts the choice of $w$. That is, we may assume that $\ell = (n+1)/2$. In particular, it follows that $n$ is odd, and that the fully weighted toughness of $P_n$ in this case is the solution to $\tau_2 = \tau_\ell$, or equivalently

$$\alpha = \frac{1}{\ell-1 + \ell+1} = \frac{1}{2A(\frac{n-2}{2})};$$

that is, the solution to $A\left(\frac{n-1}{2}\right) - \alpha = \frac{1}{2}$.

Case 2: The optimal weighting is only achieved when $r = \ell + 1$. Suppose that $\ell \neq n/2$. Without loss of generality we may assume $\ell > n/2$. Then by the above formulas, we have $w_{\ell-1} > w_{\ell+1}$.

We claim one can decrease the weight $w_r$ from its 1 without decreasing $\tau_w$. This change affects only the two ratios $\tau_\ell$ and $\tau_r$. The former will increase. For the latter, we note that for $\varepsilon > 0$ sufficiently small, $(1 - \varepsilon)/(1 + w_{\ell+1}) > 1/(1 + w_{\ell-1})$ since $w_{\ell-1} > w_{\ell+1}$. That is, the new $\tau'_w(G)$ is at least the old $\tau_w(G)$ if $\varepsilon$ is small enough. Such a weighting has a unique maximum weight, which contradicts being in Case 2.

Thus we may assume that $\ell = n/2$. In particular, it follows that $n$ is even, and that the fully weighted toughness of $P_n$ in this case is the solution to

$$\alpha = \frac{1}{1 + \ell-1} = \frac{1}{1 + A\left(\frac{n-2}{2}\right)};$$

that is, the solution to $A\left(\frac{n-2}{2}\right) = 1$.

Hence the upper bound on $\text{MFWT}(P_n)$ follows. It can be achieved by assigning weights according to the formulas. QED

### 3.3 Trees

We show here that if a tree has at least three vertices, then there is a weighting such that the fully weighted toughness exceeds the ordinary toughness. Recall that the ordinary toughness of a tree is $1/\Delta$ where $\Delta$ is the maximum degree.

**Theorem 6** If $T$ is a tree of order at least 3, then $\text{MFWT}(T) > \tau(T)$. 

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PROOF. We already know this fact for the star. (By Lemma 2 the maximum is 1, but the ordinary toughness is $1/(n - 1)$.) So assume the tree is not a star. By Theorem 2 we need only consider singleton cut-sets. We will construct a weighting $w$ such that $w(v)/m_w(T - v) > 1/\Delta$ for each cut-vertex $v$.

We define a partition $A_0, A_1, A_2, \ldots$ of $V(T)$ as follows. Let $M$ be the set of vertices of maximum degree. Define $A_0 = V(T) - M$. Define $A_1$ to be the $M$-extremal vertices, $A_2$ to be the $(M - A_1)$-extremal vertices, and so on. It follows that for $i \geq 1$, for each vertex $v \in A_i$ there is one component of $T - v$ that contains all of $(A_i - v) \cup A_{i+1} \cup A_{i+2} \cdots$. Figure 5 gives an example tree, where the numbers inside each vertex give the index $i$ of the $A_i$ to which that vertex belongs.

![Figure 5: A weighted tree](image)

Now, let $\varepsilon = 1/\Delta$. For each $i \geq 0$, assign each vertex of $A_i$ the weight

$$\mu_i = 1 + \varepsilon - \frac{\varepsilon}{2^i}.$$  

Note that every vertex receives a weight that is at least 1 but less than $1 + \varepsilon$. An example weighting is given Figure 5.

Let $\tau_i$ denote the minimum of the ratio $w(v_i)/k_w(T - v_i)$ over all $v_i \in A_i$. The numerator is always $\mu_i$. For each $v_0 \in A_0$, we have $k_w(T - v_0) < (\Delta - 1)(1 + \varepsilon)$ since $v_0$ has degree at most $\Delta - 1$. Thus

$$\tau_0 > \frac{1}{(\Delta - 1)(1 + \varepsilon)} = \frac{\Delta}{\Delta^2 - 1} > \frac{1}{\Delta}.$$  

For $i \geq 1$, for $v_i \in A_i$ we have $k_w(T - v_i) < (\Delta - 1)\mu_{i-1} + (1 + \varepsilon)$, since $v_i$ is extremal with respect to $M - A_{i-1}$, and so all components except one have
maximum vertex weight at most $\mu_{i-1}$. By the definition of the $\mu_i$, it holds that 
$2\mu_i - \mu_{i-1} = 1 + \varepsilon$. Thus 
\[
\tau_i > \frac{\mu_i}{(\Delta - 1)\mu_{i-1} + (1 + \varepsilon)} = \frac{\mu_i}{(\Delta - 2)\mu_{i-1} + 2\mu_i} \geq \frac{\mu_i}{\Delta \mu_i} = \frac{1}{\Delta}.
\]
That is, we have shown that $w(v)/k_w(T - v) > 1/\Delta$ for all choices of cut-vertex $v$. QED

4 Further Questions

For further research, it would be interesting to gain further insight into which graphs have $\text{MFWT}(G) = \tau(G)$ and which do not. Also, calculations and bounds for specific families of graphs would be helpful. For algorithmic questions, one can use similar ideas to those in [12] to calculate fully weighted toughness for interval graphs; but are there other classes where calculating $\tau_w(G)$ is polynomial? Also we have no idea on the complexity of calculating $\text{MFWT}(G)$. In another direction, this concept suggests that maybe one should investigate the weighted versions of other measures of the vulnerability of a graph.

References


