More Games

4.1 The First Player Advantage

In this section we look at a family of games called ***achievement games***. In these games there is a race to see who can first accomplish something; tictactoe is an example. Most people know that playing first is best in Tictactoe: for much the same reason, achievement games in general favor the first player.

There is a simple argument that, under certain conditions, achievement games favor the first player. Indeed, in these games, the first player, playing properly, cannot lose.

Let's try to be a bit more precise. We consider an achievement game where the players alternate selecting from some set. **All** elements of the set are available at the start. The moves and the target are the same for both players: the first player to accumulate a given subset is the winner. This is the case in Tictactoe.

The claim is the first player cannot lose. Well, suppose the second player has a winning strategy. Then all the first player needs to do is follow the rules for the second player.

It is a different story if each move changes the set of available moves. And indeed in chess and many other board games, it is clear that there are positions where the next player to move loses.

4.2 Nim

Another popular category of two-person games are what I like to call ***last straw*** games. In these games, the players alternate taking turns, and the first player unable to move loses. Again we assume there is no randomness. Two examples are the game Finger, and the game of Nim.

Last straw games seem to be harder than achievement games, both for humans and computers. The game tree is not easily prunable: if we look at a game of finger half-way through, it is still unclear who has the advantage. Sometimes in a last straw game we can see that one player has the advantage in that they have more "spare" or noncommittal moves.

Sometimes we can use symmetry to shrink the tree. For example, in Finger, the result of Anand's first move is always for Betsy to have one finger on one hand, and the result of Betsy's first move is for Anand to have one finger showing.

But every now and again, a last straw game is suddenly solvable. This isn't a really precise term, but what we mean is that there is a clear quick rule to determine the winner and the winning strategy. The grand-daddy of them all is Nim. We defined it earlier as having three piles, one with 3 coins, one with 5 coins, and one with 7 coins; but one can play the game with other starting configurations.

The game with just two piles is easy to play. If the two piles are the same, you lose. Whatever you take out of one, the other person takes out of the other. This is called a ***mirror strategy***. If the two piles are different, then to win, you must reduce the bigger pile so that the two piles are the same.

But the way to play three piles is not so easy. Consider April and Boris playing with three piles 1, 2, and 3. This is a losing position for the first player. Whatever she does, the second player can respond to leave two equal piles.

There is a famous formula on how to win. Split each pile into powers of 2 (1,2,4,8,etc). For example, the pile with 5 splits into a 1 and a 4. (It is always possible to split a number into distinct powers of 2: find the biggest power that will fit, and what is left must be less than this power, otherwise we could double the power.)

Then:

> ***Theorem.*** *A position is a losing position if and only if every power of two occurs an even number of times.*

For the original game $(3, 5, 7)$, the split is $(1 + 2, 1 + 4, 1 + 2 + 4)$. The only power that occurs an odd number of times is the 1. So you remove 1 (any one 1): the first player can win by removing one coin from any of the piles (and only that way). One must mentally rearrange piles.

One way to do the arithmetic is to write each pile size in binary. Then take the XOR of the values:

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4 2 1
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0 1 1
1 0 1
1 1 1
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0 0 1
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If you want a challenge, try playing the Miseré version of this game: the player who takes the last coin is the loser. While this game is solved, the strategy is not as easily done.

4.3 More Symmetry Arguments

There's a game:

> Coin Pack. *Take some flat surface with sides, such as a book. Then players take turns placing coins on the surface such that no coins overlap. The first player unable to place a coin is the loser.*

The first-player is the winner on this game by a mirror strategy. She places a coin exactly in the center and then copies the second player's move diagonally opposite.

A famous game was introduced by Gale.

> Chomp. *Start with a rectangular array of counters. A turn in this game is to take a counter, and all counters that are above the counter, to the right of the counter, or both. This time the person who takes the last counter is the loser.*

Let's warm up with a 2-by-2 grid. It is easy to check (come on now, you can do it), that the first player wins this game.

Now for the curious part (and very annoying to mathematicians). This game should always be won by the first player. (Remember we can, given enough time and space, write down the game tree, compute the winner, and so somebody has a winning strategy.)

The reasoning is the following. Consider the smallest possible first move: taking just the coin in the top-right corner. There are two possibilities. The first possibility is that it is a winning move in the game tree. Okay then, the first player wins.

The second possibility is that the smallest move loses. So there is some reply that wins for the second player. Well, all the first player needs to do is to play that response first. And then follow the second player's winning strategy. So in either case, the first player actually can win. (Okay I admit this is slippery.)

But the annoying thing is that this argument, slick as it is, gives us no information as to what is the winning strategy. And the game remains unsolved: nobody has a quick formula or algorithm to work out even the winning move (or even to answer the simpler question of whether the smallest move first wins or loses).

Exercises

4.1. Who wins the Miseré version of Nim starting with piles of 1, 2 and 5?

4.2. Determine whether the 1-counter move is a winning move in the 3-by-3 version of Chomp.

4.3. Show that a mirror strategy can sometimes be used in Dawson's Kayles.