Simultaneous Games

Here the two players play simultaneously. The classic example is found in most cultures under various names.

> ROCK-PAPER-SCISSORS. *Players simultaneously choose one of Rock, Paper or Scissors. Rock beats Scissors, Scissors beats Paper, Paper beats Rock.*

Everybody knows this is random, but you can try to outguess, outpsyche your opponent. There is even a Rock-Paper-Scissor world championship.

5.1 The Foxhole Problem: An example

> THE FOXHOLE PROBLEM. *Your opponent has 5 foxholes in a row to shelter in. You can lob one mortar to hit and destroy two adjacent foxholes. Which pair should you aim at?*

We can associate a payoff with each result. Maybe, from your perspective, there is a payoff of +1 for a hit, and 0 for a miss. You try to maximize the payoff; your opponent tries to minimize it. The game can be summarized in the following payoff matrix.

		Hider				
		1	2	3	4	5
	1–2	1	1	0	0	0
Firer	2–3	0	1	1	0	0
	3–4	0	0	1	1	0
	4–5	0	0	0	1	1

Well, the hider can ensure at most 1/3 chance of a hit, by hiding at random in either the first, middle or fifth hole (only one of which can be touched by a single mortar). The firer can ensure at least a 1/3 chance of a hit with the following approach: label the first pair 1,2, the next hole 3, the next hole 4, and the last pair 5,6, and then rolling a dice. That is, he picks an end with probability 1/3, and a middle hit with probability 1/6. Thus we say that the value of the game is 1/3. This behavior is typical:

> ***The Minimax Theorem.*** *For any 2-person zero-sum simultaneous game, there exist a value v and strategies for both players such that: the first player's strategy guarantees her at least v expected return and the second player's strategy guarantees him at least $-v$ expected return.*

The strategies are always to play each option with a certain probability. The strategies and the value of the game can be calculated by linear programming.

5.2 Solving Some Games

In some special circumstances, the game can be solved without resorting to linear programming. If we know for sure that every option will be used, we can choose a strategy that gives the same expected value regardless of what the other player plays.

Consider the Battle of Midway problem between Rosie the row-chooser and Colin the column-chooser where the chance of success (from Rosie's perspective) is given by the following table:

	North	South
West	0.3	0.1
East	0	0.5

We know that Rosie should play each row with some probability. Say she plays row West with probability p. Then if Colin plays column North, Rosie's chance of success is $0.3p$; if Colin plays column South, her change of success if $0.1p + 0.5(1-p)$. Setting these two equal, we solve for p and obtain $p = 5/7$. Similarly, one can calculate that Colin should play column North with probability $q = 4/7$.

5.3 Final Jeopardy

At the end of the Jeopardy television show comes Final Jeopardy. Here each player stakes all or part of their winnings on answering one question, after which the player with the highest total is the champion. Many players, especially the leaders, seem unaware that this betting stage is a classic simultaneous game.

Let's do a simplified version. Say only two contestants remain at the start of Final Jeopardy: Amy with $10,000 and Bart with $7,000. Suppose first that Amy bets $4,001 or more. Then she is guaranteed to win, provided she answers the question correctly. Assuming the goal is for her to maximize her chance of being champion, she should then worry about what happens when she gets the question wrong. In which case she should bet as little as possible. All of this logic is to argue that she should not bet more than $4,001. And no leader ever does.

Now consider Bart. If he knows for a fact absolutely and utterly that Amy will bet $4,001, then he should bet nothing, guaranteeing him a win if she answers wrong. But if Amy knows for a fact that Bart bets nothing, then she should follow suit, guaranteeing her a win. And yet if Bart knows Amy is betting nothing, he should bet $3,001 so that he is champion if he answer correctly. How to get out of this circular logic?—realize it's a simultaneous game.

So consider Bart again. Suppose that he bets more than $1,001. If he gets the final question wrong, then he is guaranteed to lose—Amy will always have at least $5,999. So he should worry about what happens when he gets the question correct. In which case, he should bet as much as possible. All of this logic is to argue that: IF Bart bets more than $1,001, THEN he should bet everything. This maximizes his chances of being champion. And we often see the trailing player bet everything.

So, now comes the interesting part. We have proven that Bart should either bet at most $1,001 or bet everything. Suppose now that Amy bets less than $4,000. Then if Bart bets everything, his answer determines the outcome: Amy's response and the exact value of her small bet are irrelevant. So she should worry about what happens when he makes a small bet. In which case she can maximize her chance of being champion by betting anything under $1,999, for example, nothing.

And so Bart can bet on the assumption that Amy bets either $4,001 or nothing. In which case, it matters not one bit the exact value of his small bet, so we can suppose it is $1,000. So we have a theorem:

> ***Theorem.***
> *Amy has an optimal strategy which plays only $4,001 or nothing.*
> *Bart has an optimal strategy which plays only $7,000 or $1,000.*

We have a table that says when Amy wins:

		Bart	
		Low	High
Amy	Low	always	if Bart is wrong
	High	if Amy is correct	unless Amy is wrong and Bart is correct

Note that even if Amy publicly reveals her strategy (but not which of the two she plays), Bart cannot improve on his strategy. And likewise Amy cannot take advantage if Bart reveals his general strategy. The exact proportions each player should play each bet depends on the probabilities of each player getting the question correct. See the exercises.

5.4 Other Simultaneous Games

As soon as you remove the zero-sum nature of the game (where one player's loss is exactly the other player's gain), things are not so easy. For example, the famous prisoner's dilemma:

> PRISONER'S DILEMMA. *You and a colleague are stuck in a Martian jail without a means of communication. You are being tortured. You can either confess or not confess. If you both confess, you'll both be killed immediately. If you confess and your partner doesn't, you'll be released immediately by turning state's evidence; but if you don't confess and your partner does, you'll be killed, slowly. If neither confesses, then the Martians will get tired and release you, after a while.*

Suppose your colleague keeps quiet. Then you are better off to confess: you go free immediately. Suppose your colleague confesses. Then you are better off to confess, since it'll mean a merciful death. So you should both confess? But both not confessing is better than this.

Nash (as in the movie Beautiful Mind) pioneered research on the existence of rational strategies for such games.

And things also change dramatically if there are more than two players. There is always the scope for collusion, even if the game is zero-sum.

Exercises

5.1. In the game of Thumbs, there are two players: Matcher and Opposer. Both players simultaneously produce either one thumb or two. If the number of thumbs are the same, the Matcher wins the total number of thumbs in dollars from the other player. If the number of thumbs is difference, the Opposer wins the total number of thumbs in dollars. The game from the Matcher's perspective as row-player:

	Two	One
Two	$+4$	-3
One	-3	$+2$

(a) What is the optimal strategy for the Matcher?

(b) Which player would you prefer to be? Why?

(c) What should Opposer do if she knows she is playing against a player who always plays each number with equal probability?

5.2. Consider the scenario from Final Jeopardy discussed above. Suppose that there is a 2/5 chance they both get the question correct, a 1/5 chance that Amy gets the question correct while Bob does not, a 1/5 chance that Bob gets the question correct while Amy does not, and a 1/5 chance they both get it wrong.

(a) Construct the payoff matrix for the game, where the number in the cell is the probability that Amy is champion.

(b) Calculate the proportions each player should play each option, and the overall chance of Amy being champion.