2 More Counting

2.1 Unordered Sets and Binomial Coefficients

In counting sequences, the ordering of the digits or letters mattered. Another common situation is where the order does not matter, for example, if we want to choose a subset of a given size.

Example 2.1. Suppose we have 4 players, say A, B, C, D, but only two can proceed. How many possible pairs are there?

Well, there are 6 possible pairs: AB, AC, AD, BC, BD, and CD. Note that the order within the pair does not matter: AB is the same as BA.

This is the binomial coefficient’s job. The answer we want is abbreviated \( \binom{4}{2} \). Some people write this using a capital C, such as \( 4C_2 \), but we will not.

Lemma 2.1 Given a universe \( X \) of \( n \) elements, the number of ways to choose an unordered subset of \( X \) of \( k \) elements of without replacement (assuming \( 0 \leq k \leq n \)) is the binomial coefficient

\[
\binom{n}{k} = \frac{n!}{k! (n-k)!}
\]

pronounced “\( n \) choose \( k \).”

Proof. One way to prove the formula is to argue as follows. Let \( A \) be the number of ordered sequences of \( k \) elements (without replacement) and let \( B \) be the number of unordered subsets of \( k \) elements (without replacement).

We already calculated \( A \) in Lemma 1.2 as being \( n!/(n-k)! \). But we can also generate all ordered sequences on length \( k \) in the following way. Generate all unordered subsets of size \( k \). There are \( B \) of these. Then order each of these subsets in all possible ways. We know that a subset of size \( k \) can be ordered in \( k! \) ways. So this means that:

\[
A = B \times k!
\]

It follows that \( B = n!/(k!(n-k)!) \), as the lemma claims. 

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This lemma can also be argued in reverse. If we write out all ordered sequences, and group them according to the subset to which they correspond, then each subset will appear $k!$ times. For example, suppose $n = 4$ and $k = 3$. Then there are $4!/(4 - 3)! = 24$ sequences, but only 4 subsets:

- $\text{abc, acb, bac, cab, cba} \rightarrow \{a, b, c\}$
- $\text{abd, adb, bad, dab, dba} \rightarrow \{a, b, d\}$
- $\text{acd, adc, cad, dac, dca} \rightarrow \{a, c, d\}$
- $\text{bcd, bdc, cbd, dbc, dcb} \rightarrow \{b, c, d\}$

One obvious property from the symmetry of the formula is that:

**Lemma 2.2**

$$\binom{n}{k} = \binom{n}{n-k}.$$

This fact can also be observed by noting that choosing the $k$ elements in the subset is equivalent to choosing the $n - k$ elements not in the subset.

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**Example 2.2.** Suppose there is a league of 10 teams and every team must play every other team exactly once. How many matches are there?

The answer is $\binom{10}{2} = 45$.

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**Example 2.3.** How many 5-letter words are there with exactly two vowels?

Start by choosing the places where the vowels will go. This is 2 places out of 5, so there are $\binom{5}{2} = 10$ ways to do this. Then choose the vowels—$5^2$—and the consonants—$21^3$. So the answer is $10 \times 5^2 \times 21^3$.

The above example illustrates a common approach—choose the pattern, then choose the way to fill the pattern.

▶ For you to do! ◀

**At the local ice-cream parlor there are 20 mixins, and you get to pick 3 different mixins. Except that 4 of the mixins are “premium” and you cannot have more than one premium mixin. How many possibilities are there? (Hint: there are two choices for the number of premium mixins used.)**
2.2 More Examples

Time for some more examples.

**Example 2.4.** Consider a bag with 4 identical amber balls, 4 identical blue balls, and 4 identical carmine balls. In how many ways can I:

(a) Pick a subset of the balls?
(b) Pick a subset of 4 balls so that I have at least one ball of each color?
(c) Pick a subset of 7 balls so that I do not have at least one ball of each color?

(a) Because the balls of the same color are identical, it is only the number of each color that matters. There are five possibilities for the number of amber balls (0, 1, 2, 3, 4) and similarly for each of the other colors. So the answer is $5^3$.

(b) The only choice we get is which color is doubly represented. Answer is 3.

(c) The subset must have 4 balls of one color and 3 balls of another color. So we get to choose the majority color and then the minority color. The answer is $3 \times 2 = 6$.

**Example 2.5.** How many anagrams of TATTERED including nonsense words are there?

Well, there are 8 letters. But this does not mean there are 8! anagrams, since the order of the T’s for example does not matter. (Consider for instance the situation where all the letters are the same!) Note that there are 3 T’s and 2 E’s, and the remaining letters each appear once. The claim is that the answer is

$$\frac{8!}{3! \times 2!}.$$

One way to see this, is to temporarily add subscripts to the T’s and E’s. Then list all anagrams: there are indeed 8! if we consider the three T’s and two E’s as distinct. But if we now erase the subscripts, then a word like TAREDETT will appear $3! \times 2! = 12$ times on our list—there are 3! ways of arranging the subscripts on the T’s and 2! ways of arranging the subscripts on the E’s. So our 8! anagrams can be arranged into groups of 12 identical words. That is, there are $8!/12 = 3360$ anagrams.

**Lemma 2.3** If we have letters $L_1, L_2, \ldots, L_k$ with counts $c_1, c_2, \ldots, c_k$, then the number of anagrams of all the $T = c_1 + c_2 + \ldots + c_k$ letters combined is

$$\frac{T!}{c_1! \times c_2! \times \cdots \times c_k!}.$$
The proof is a repetition of the argument in the above example.

The following example talks about the odds of an event. The odds or chance of an event is the likelihood of it occurring, or the proportion of time that it occurs. This is calculated by dividing the number of outcomes corresponding to the event in question by the total number of outcomes. (This assumes all outcomes are equally likely; for example, it assumes that the pack of cards is perfectly shuffled.)

Example 2.6. In poker, what are the odds of being dealt a “flush” (all cards are the same suit)? Let’s assume we have a standard 52-card deck, nothing wild, and are dealt 5 cards.

Well there are \( \binom{52}{5} \) possible hands. A calculator shows this is 2598960. There are 4 possibilities for the suit. For a fixed suit, there are \( \binom{13}{5} \) ways to deal a flush with that suit. Thus the total number of flushes is \( 4 \times \binom{13}{5} = 5148 \). The proportion/chance is 0.198%, or about 1 time in 505.

▶ For you to do! ◀

1. To buy a ticket in the local PowwowBall lottery you choose 6 numbers from the range 1 to 54 as well as a single Powwow ball in the range 1 to 42. How many different tickets are there?

2.3 Multisets

Binomial coefficients turn up in maybe unexpected places.

A multiset is like a set except elements can be repeated. One might view it as an “unordered subset with repetition allowed”.

Example 2.7. Determine the number of 3-element multisets of \( \{a, b, c\} \).

There are 10 multisets: aaa, bbb, ccc, aab, aac, bba, bbc, cca, ccb, abc.

Example 2.8. Determine the number of 3-element multisets of an \( n \)-element set.

There are three patterns. all three elements the same: \( n \) choices; two of one and one of another: \( n \times (n - 1) \); and all three different: \( \binom{n}{3} \).

By arithmetic, this sums to \( \binom{n+2}{3} \). (Check this yourself!)

The fact that the above example gives a binomial coefficient is no coincidence.
Lemma 2.4 Given a universe of $n$ elements, the number of ways to choose an unordered multiset of $k$ elements (assuming $0 \leq k$) is

$${\binom{n+k-1}{k}}.$$

We will prove this in a moment. (And I agree in advance that the proof is intimidating—so you might want to skip it.) But here is an application

**Example 2.9.** Suppose we roll two dice. If the dice are indistinguishable, how many outcomes are there?

We already calculated this in Example 1.6. Counting dice throws with indistinguishable dice is equivalent to counting unordered multisets. That example corresponds to $n = 6$ and $k = 2$ in the above lemma. The lemma says that the number of outcomes is $\binom{7}{2} = 21$, which is what we determined originally.

Okay, here is the proof of the lemma.

**Proof of Lemma 2.4.** We will count something that is the same size.

Fix some ordering of the $n$-element universe $X$. The key idea is that each $k$-element multiset $S$ from $X$ can be represented by a sequence of $k + n - 1$ white and black circles of which $k$ are white, and vice versa.

The recipe is to write $S$ on a line. Then insert a black circle for every change in $X$. Then convert every element of $S$ to a white circle.

For example, suppose $X = \{a, b, c, d, e, f\}$ and consider $\text{aacd}$. Then we have

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a  a  c  d

a  a  ●  ●  c  ●  d  ●  ●

○  ○  ●  ●  ○  ●  ○  ●  ●
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Now, the claim is that this process is reversible; That is, we can get from the circles to the multisets. We start with $a$ and every time we hit a dark circle we increment it. Furthermore, and this is the magical thing, every possible sequence of circles with $k - 1$ blacks corresponds to a multiset.

And so, counting the number of multisets is the same as counting the number of circle sequences. And there are $\binom{n+k-1}{k}$ of those. ♦
Example 2.10. Suppose we have 7 numbered boxes in a row. We have 10 indistinguishable balls. In how many ways can we place the balls into the boxes?

If we write down the labels of the boxes taken, this problem is equivalent to choosing a 10-element multiset from a 7-element universe. By the above lemma, this is the binomial coefficient \( \binom{10}{10} \).

The lemma explains the answer to the question of unordered subsets with repetition allowed.

**Exercises**

2.1. A contest has 5 prizes: TV, ipod, cellphone, bicycle, vacation. You get to pick 2 of them, except that you cannot pick two electronic goods. How many possibilities do you have?

2.2. Calculate \( \binom{7}{1} \), \( \binom{4}{2} \), \( \binom{6}{3} \), \( \binom{8}{4} \), and \( \binom{10}{5} \).

2.3. An SSN is a 9-digit number with zeroes allowed in every position.

   (a) How many SSNs have exactly two distinct digits?
   (b) How many SSNs have digits that sum to 2?
   (c) How many SSNs have digits that sum to 3?

2.4. How many 5-letter words are there with exactly 4 different letters?

2.5. Determine the number of anagrams of:

   (a) SASSAFRAS
   (b) BOOKKEEPER

2.6. A pizza place offers 5 different meat toppings, and 10 different vegetable toppings. (In each of the following the order of the toppings does not matter.)

   (a) A meat-and-three pizza has 1 meat and 3 different vegetable toppings. How many meat-and-three pizzas are there?
   (b) A double-play has only 2 toppings, but these can be the same topping. How many double-play pizzas are there?
   (c) A glutton pizza has 6 different toppings of which at most 4 can be meat. How many glutton pizzas are there?
2.7. I have a bag containing 12 numbered balls of which 4 are red, 4 are green, and 4 are blue. In how many ways can I choose an unordered set of:

(a) 7 balls?
(b) 6 balls if I must have equal numbers of each color?
(c) 4 balls if I must have at least one of each color?
(d) 4 balls if I must have more red than green?

2.8. Let us define a **key** as a 6-digit number with zeroes allowed in every position, such as 043771.

(a) How many possible keys are there with all digits distinct?
(b) How many possible keys are there if the digits sum to 3?
(c) How many possible keys are there if the key contains exactly two distinct digits?

2.9. Consider three balls and three buckets. In how many different ways can the balls be arranged in the buckets if:

(a) the balls and the buckets are all numbered?
(b) the balls are numbered but the buckets are indistinguishable?
(c) the buckets are numbered but the balls are indistinguishable?
(d) the balls are indistinguishable and the buckets are indistinguishable?

2.10. Wayne has a pile of 20 books to read.

(a) In how many orderings can he read them?
(b) In how many orderings can he read them if the pile includes the 7 Harry Potter books, which must be read in order and consecutively?
(c) How many orderings if the 7 Harry Potter books must be read in order but not necessarily consecutively?

2.11. Suppose I have a bag with $X$ balls labeled 1 up to $X$, with $X$ even. Half the balls are orange and half the balls are purple.

(a) In how many ways can I choose a subset of 3 balls such that their labels sum to at most 8? (Assume $X$ is large.)
(b) In how many ways can I choose a subset of 4 balls such that I get at least one ball of each color.

2.12. Jabber is played with a 30-card deck: there are three suits and the cards are numbered 1 up to 10. A player receives 4 cards.
(a) How many possible jabber hands are there?
(b) A straight contains cards of consecutive values, such as 5, 6, 7 8, but they can be of different suits. How many possible straights are there?
(c) A flush has all cards the same suit. How many possible flushes are there?
(d) A straight flush is a hand that is both a straight and a flush. How many possible straight flushes are there?

2.13. In the local lottery, you buy a ticket with 6 (unordered) numbers in the range 1 to 49, and you have to match the 6 numbers drawn to win the jackpot.

(a) Calculate \( \binom{49}{6} \).
(b) A runner-up prize is obtained if you match exactly 5 of the drawn numbers. Calculate the odds of a runner-up prize.
(c) All tickets that match no numbers are placed in a barrel for a chance at a “lucky loser” prize. Calculate the odds of a particular ticket matching no numbers.


(a) Calculate the odds of being dealt a “full house” (3 of one denomination/rank and 2 of another).
(b) Calculate the odds of being dealt a “royal flush” (ace, king, queen, jack, and ten of the same suit).
(c) Calculate the odds of being dealt “two pairs” (2 of one denomination, 2 of another denomination, and the remaining card a third denomination).

2.15. Consider 3-digit numbers, no zeroes allowed. How many such numbers are there with the digits strictly increasing?

2.16. Mabe lives in Manhattan and his office is 5 blocks east and 3 blocks north. He always takes the shortest route to work (that is, he walks exactly 8 blocks), but he likes to vary the route.

(a) How many different shortest routes can Mabe take between his home and his office?
(b) How many different shortest routes can Mabe take if he wants to walk along two sides of the central green?

(c) How many different shortest routes can Mabe take if he wants to avoid the central green completely (even at its corners)?

2.17. A bracelet is defined to have no starting point, and flipping it over gives the same bracelet. (For example, if I have 4 beads, 2 orange and 2 purple, then I can make two different bracelets.) How many different bracelets can be made with 5 beads: 2 red, 2 blue, and 1 white?

2.18. Suppose we have 12 people and need to split them into three subcommittees of size 3, 4 and 5, with nobody serving on more than one subcommittee. In how many ways can this be done?

2.19. On Planet X all the people are of the same gender. Nevertheless, they still pair off each year to get married in one simultaneous ceremony. There are 10 people on planet X. How many possible marriage ceremonies are there?

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Solutions to Practice Exercises

1. You get either three non-premiums or two non-premiums and one premium. Thus the answer is \( \binom{16}{3} + \binom{16}{2} \cdot \binom{4}{1} \).

2. Answer is \( 42 \times \binom{54}{6} \).