11 Mathematical Induction

The principle of **mathematical induction** rests on the following idea. Assume \( p(n) \) is some predicate about an integer \( n \):

**Mathematical Induction.** If statement \( p(b) \) is true, and

\[
statement \; p(n - 1) \Rightarrow p(n) \; \text{is true for all } n > b,
\]

then \( p(n) \) is true for all integers \( n \geq b \).

The justification is that by the first part we have that \( p(b) \) is true. Then by the second part we have that \( p(b + 1) \) is true. Then by the second part we have that \( p(b + 2) \) is true. And so on.

There is a standard recipe for proofs using mathematical induction. We first prove the **base case.** Then we assume \( p(n - 1) \) is true—called the **inductive hypothesis** and abbreviated IH—and using this fact, we prove that \( p(n) \) is true. In this course, many of our examples are proving formulas. In this case, we use LHS to stand for the left-hand side, and RHS for the right-hand-side.

**Example 11.1.** Prove that \( 1 + 2 + 3 + \ldots + n = n(n + 1)/2 \) for all \( n \geq 1 \).

**Base case:** When \( n = 1 \), LHS = 1; RHS = 1(1 + 1)/2 = 1; so LHS = RHS.

**Inductive step:** Assume formula true for \( n - 1 \); show for \( n \). Then

\[
\begin{align*}
\text{LHS} &= 1 + 2 + 3 + \ldots + n \\
&= [1 + 2 + 3 + \ldots + n - 1] + n \\
&= \frac{(n - 1)((n - 1) + 1)}{2} + n \; \text{by IH} \\
&= \frac{(n - 1)n + 2n}{2} \\
&= \frac{n(n + 1)}{2} \\
&= \text{RHS}
\end{align*}
\]

Thus the formula is true for all \( n \geq 1 \).

When proving formulas for sums, we often use this “peeling” idea; that is, we take the whole sum and separate out the part for \( n - 1 \).

**Example 11.2.** Prove by induction for all \( n \geq 0 \), that \( \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \).
Base case: When \( n = 0 \), \( \text{LHS} = 2^0 = 1 \); \( \text{RHS} = 2^1 - 1 = 1 \); so \( \text{LHS} = \text{RHS} \).

Inductive step: Assume formula true for \( n - 1 \); show for \( n \). Then

\[
\text{LHS} = \sum_{i=0}^{n} 2^i = 2^{n-1} + 2^n \quad \text{by IH} = 2^n - 1 + 2^n = 2^{n+1} - 1 = \text{RHS}
\]

Thus the formula is true for all \( n \geq 0 \).

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**Example 11.3.** Prove that \( 2^{n+1} > n^2 + 3 \) for all \( n \geq 2 \).

Base case: When \( n = 2 \), \( \text{LHS} = 2^3 = 8 \), and \( \text{RHS} = n^2 + 3 = 7 \), so statement true for \( n = 2 \).

Inductive step: Assume \( n > 2 \) and statement true for \( n - 1 \); that is, \( 2^n > (n - 1)^2 + 3 \). We want to show that \( 2^{n+1} > n^2 + 3 \). Then

\[
\text{LHS} = 2^{n+1} = 2 \times 2^n > 2 \times ((n - 1)^2 + 3) \quad \text{by IH} = 2(n^2 - 2n + 4) = 2n^2 - 4n + 8 = (n^2 + 3) + ((n - 2)^2 + 1) > n^2 + 3 = \text{RHS}
\]

as required. Therefore, the result is true by mathematical induction.

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**Example 11.4.** Show that \( 2^n + 3^n \) is a multiple of 5 for all positive odd integers \( n \).

Base case: When \( n = 1 \), \( 2^1 + 3^1 = 5 \), which is a multiple of 5.

Inductive step: Let \( n = 2i + 1 \). Assume true for \( i - 1 \), test for \( i \). Then

\[
2^{2i+1} + 3^{2i+1} = 4 \times 2^{2i-1} + 9 \times 3^{2i-1} = 4 \times (2^{2i-1} + 3^{2i-1}) + 5 \times 3^{2i-1}.
\]
The first term is a multiple of 5 by the IH. The second term is a multiple of 5 too. Therefore the sum is a multiple of 5. Hence $2^n + 3^n$ is a multiple of 5.

In **strong induction**, the inductive step is replaced by the following:

$$p(b) \land p(b + 1) \land \cdots \land p(n - 1) \Rightarrow p(n)$$

But we do not need that here.

▶ For you to do! ◀

1. Prove that $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n - 1) \times n = \frac{n^3 - n}{4}$ for all $n \geq 2$.

**Exercises**

11.1. Prove that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ for all $n \geq 1$.

11.2. Prove by induction that for all $n \geq 1$

$$\sum_{i=1}^{n} i \cdot 2^i = (n - 1)2^{n+1} + 2.$$ 

11.3. Prove by induction that for all $n \geq 1$:

$$\sum_{i=1}^{n} \frac{1}{i(i + 1)} = 1 - \frac{1}{n + 1}$$

11.4. Prove that $n! > 2^n$ for $n \geq 4$.

11.5. Prove by induction that $\sum_{i=1}^{n} i^2 = n(n + 1)(2n + 1)/6$.

11.6. Prove by induction a formula for $\sum_{i=1}^{n} i^3$. (Hint: it is a polynomial of degree 4.)

11.7. Prove by induction that for all $n \geq 1$

$$\sum_{i=1}^{n} (-1)^i \cdot i^2 = \frac{(-1)^n n(n + 1)}{2}$$

(Note that the sum starts $-1 + 4 - 9 + 16 \ldots$)

11.8. Winifred wants the formula for

$$\sum_{i=1}^{n} i^2 2^i$$

(That is, $1 \times 2^1 + 4 \times 2^2 + 9 \times 2^3 + 16 \times 2^4 \ldots$) She finds the formula in a book, except that someone has smeared a bit of ink on it. The result is that the formula reads

$$(n^2 - 2n + \oplus)2^{n+1} - \ominus$$

Determine what the two $\oplus$’s are. (Hint: each is a (different) positive integer.)
11.9. Prove by induction that $3^n \geq n^3$ for all integers $n \geq 3$.

11.10. Prove that $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ for all integers $n \geq 1$.

11.11. Prove by induction that for every nonnegative integer $n$, $4^{2n+1} + 11^n$ is divisible by 5.

11.12. Prove by induction that for every nonnegative integer $n$, $6^{2n+1} + 4^{3n}$ is divisible by 7.

11.13. Prove that the sum of the interior angles of an $n$-gon is $(n-2)180$ degrees.

11.14. Prove that $\sum_{i=1}^{\infty} \frac{1}{i}$ is infinite. (Hint: show by induction that $\sum_{i=1}^{2k} \frac{1}{i} \geq k/2$.)

11.15. Prove by induction that there are at least $n$ primes for all positive integers $n$. (That is, there are infinitely many primes.)

11.16. Consider a prison which is the shape of an $n$-gon (not necessarily convex). The warden has three teams of guards, a red team, a blue team, and a green team. The warden wants to assign each vertex of the polygon to one (guard of a) specific team, such that every point in the interior of the polygon is visible to (at least one member of) each team.

(a) Prove by induction that this is possible.

(b) Construct a 100-gon prison where, if we have to leave one of the vertices unassigned, then it is not possible to satisfy the warden’s requirements.

11.17. Using the Internet, write a page on the Unexpected Hanging Paradox.

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**Solutions to Practice Exercises**

1. **Base case:** LHS = $1 \times 2 = 2$; RHS = $(2^3 - 2)/2 = 2$; so LHS = RHS.

   **Inductive step:** Assume formula true for $n - 1$; show for $n$. Then

   \[
   \text{LHS} = 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n - 1) \times n \\
   = [1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n - 2)(n - 1)] + (n - 1) \times n \\
   = \frac{(n - 1)^3 - (n - 1)}{3} + (n - 1)n \quad \text{by IH} \\
   = \frac{n - 1}{3} \left[(n - 1)^2 - 1 + 3n\right] \\
   = \frac{n - 1}{3} \left[n^2 + n\right] \\
   = \text{RHS}
   \]