11 Sequences and Recurrences

A sequence is just what you think it is. It is often given by a formula known as a recurrence equation.

11.1 Arithmetic and Geometric Progressions

An arithmetic progression is a sequence where every two consecutive entries differ by the same amount. For example,

\[4, 7, 10, 13, 16, \ldots\]

If the sequence has first term \(A\) and last term \(L\) and there are \(n\) terms in the sequence, then an arithmetic progression has sum

\[\frac{n(A + L)}{2}\]

(Proof left as exercise.)

A geometric progression is a sequence where every two consecutive entries have the same ratio. For example,

\[2, 6, 18, 54, 162, \ldots\]

A useful fact is the sum of a geometric progression:

\[\sum_{i=0}^{n-1} Ar^i = \frac{A(r^n - 1)}{r - 1}\]

provided \(r \neq 1\). This can be proved by induction, or by this idea:

\[r \cdot \text{LHS} = \sum_{i=0}^{n-1} Ar^{i+1} = \sum_{i=1}^{n} Ar^i = \text{LHS} + Ar^n - A,\]

so that \((r - 1)\text{LHS} = A(r^n - 1)\).

11.2 Fibonacci Numbers

Consider a board like a checkerboard that is partitioned into squares. Define a tiling of a board to mean covering the board completely with nonoverlapping dominoes, where each domino covers two adjacent squares. Consider a board with 2 rows and \(n\) columns. Clearly, there is a tiling with all vertical dominoes. But how many tilings are there? For example, if there are 3 columns, there are two other tilings, each with only 1 vertical domino:

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Let $f(n)$ be the number of domino tilings of the $2 \times n$ board. Let’s look for a way of writing $f(n)$ in terms of smaller values. The idea is to look at the left end of the board. There are two possibilities: either there is a vertical domino at the left, or there are two horizontal dominoes. In the first case, the remaining dominoes form a tiling of the $2 \times (n-1)$ board; in the second case, the remaining dominoes form a tiling of the $2 \times (n-2)$ board. By the sum rule, we thus have:

$$f(n) = f(n-1) + f(n-2).$$

If there is only 1 column, then the recurrence formula breaks down, as two horizontal dominoes are impossible. If there are 2 columns, then the recurrence formula is valid, provided one defines $f(0) = 1$.

With some paper, one can calculate $f(n)$, starting with $f(0)$:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

These are the famous Fibonacci numbers. Gazillion things are counted by these.

They also have many patterns—here is one example.

**Theorem 11.1** For all $n \geq 1$,

$$[f(n)]^2 - f(n-1)f(n+1) = (-1)^n.$$

**Proof.** Proof by mathematical induction. The base case is $n = 1$. LHS $= [f(1)]^2 - f(0)f(2) = 1^2 - 1 \cdot 2 = -1$. RHS $= (-1)^1 = -1$. So the base case is true.

Now for the induction step. Assume the statement is true for $n-1$; we need to prove it for $n$. Well, start with the LHS for that case, use the definition of the Fibonacci sequence, and do some algebra:

$$\text{LHS} = [f(n)]^2 - f(n-1)f(n+1)$$
$$= f(n)[f(n-1) + f(n-2)] - f(n-1)[f(n) + f(n-1)] \quad \text{(by Fibonacci defn twice)}$$
$$= f(n)f(n-2) - f(n-1)f(n-1) \quad \text{(by simplification)}$$
$$= -(-1)^{n-1} \quad \text{(by the induction hypothesis)}$$
$$= (-1)^n = \text{RHS},$$

as required. ♦
The original story behind the Fibonacci numbers was the following: A pair of rabbits starts breeding after two months and produces one pair every month thereafter. Assume we start with one new-born pair of rabbits. Let \( R(n) \) be the number of pairs of rabbits after \( n \) months. Then we claim that

\[
R(n) = R(n-1) + R(n-2)
\]

with \( R(0) = R(1) = 1 \). For, we still have the rabbits we had the month before. And every pair that is at least two months old produces a new pair. The number that are two months old are the ones that were alive two months ago.

There is a weird-looking formula for Fibonacci numbers:

\[
f(n) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right).
\]

We shall see where this comes from in a moment.

### 11.3 Recurrence Equations

The formula for the Fibonacci numbers is an example of a recurrence. Here is another example.

**Example 11.1.** Find a recurrence for \( S(n) \), the number of subsets of \( \{1, 2, 3, \ldots, n\} \).

Every subset of \( \{1, 2, \ldots, n-1\} \) can be extended to a subset of \( \{1, 2, 3, \ldots, n\} \) by either adding or not adding the element \( n \). Therefore

\[
S(n) = 2S(n-1) \text{ for } n \geq 1 \text{ and } S(0) = 1.
\]

It follows immediately that \( S(n) = 2^n \).

Most recurrence relations have **initial conditions**, since the recursive formula breaks down eventually for the smallest \( n \). Note that without knowing the initial condition, the recurrence \( S(n) = 2S(n-1) \) has multiple solutions: \( S(n) = \alpha 2^n \) is a solution for any real number \( \alpha \) (including zero!). One can **verify** that some formula is a solution by plugging it into both sides and checking that one gets the same value. (Do it here!)

**Example 11.2.** Find a recurrence for \( P(n) \), the number of unordered pairs from the set \( \{1, 2, 3, \ldots, n\} \).

We saw already that \( P(n) = \binom{n}{2} \). But we can obtain a recurrence. Partition the pairs into two sets based on whether they contain the element \( n \) or not. There are \( n-1 \) pairs that contain element \( n \), and \( P(n-1) \) that don’t. So

\[
P(n) = n - 1 + P(n-1)
\]
with initial condition \( P(1) = 0 \).

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**Example 11.3.** My money \( A \) is invested at interest rate of \( p \) percent compounding annually. What do I have after \( n \) years?

Let \( M(n) \) be the amount after \( n \) years. Then \( M(n) = (1 + p/100)M(n - 1) \), with \( M(0) = A \).

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### 11.4 Some More Recurrences

Here are some more counting problems.

**Example 11.4.** For sequences of length \( n \) using letters \( A \), \( B \), and \( C \), determine a recurrence for:

(i) the count \( a(n) \) where every \( A \) is followed by a \( B \).  
(ii) the count \( b(n) \) here every block of \( B \)'s has length a multiple of 3.  
(iii) the count \( c(n) \) such that there is exactly one \( C \).

(i) If the first char is not an \( A \), then the remainder is any valid sequence. If the first char is an \( A \) then the next char must be an \( B \), but thereafter there is any valid sequence. Thus \( a(n) = 2a(n - 1) + a(n - 2) \).

(ii) If the first char is not a \( B \), then the remainder is any valid sequence. If the first char is a \( B \), then the next two chars must be \( BB \), but thereafter there is any valid sequence. Thus \( b(n) = 2b(n - 1) + b(n - 3) \).

(iii) Obviously it is better to do this without recurrences. But if the first char is not a \( C \), then the remainder is any valid sequence. If the first char is a \( C \), then the remainder is only \( A \)'s and \( B \)'s. Thus \( c(n) = 2c(n - 1) + 2^{n-1} \).

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Here is a famous recurrence. A string of parentheses is called **valid** if it is of the correct form for an arithmetic expression. That is, the parentheses pair off such that every two pairs either nest or don’t overlap at all. For example, there are two valid strings of 4 parentheses: \( (()) \) and \( ()() \). (The valid strings have the property that, reading left to right, the number of left parentheses is always at least the number of right parentheses.)

Let \( p(n) \) be the number of valid strings using \( 2n \) parentheses. This has the recurrence:

\[
p(n) = \sum_{i=1}^{n} p(i - 1) \times p(n - i) \quad (n \geq 1),
\]

with \( p(0) = 1 \).
To prove the recurrence. Consider any valid string of parentheses. Then the parentheses pair off. Consider the first parenthesis (a left one) and its partner. Suppose that it partners with the \( i \)th right parenthesis. Then the string \( A \) between these two is itself a valid string of length \( 2(i - 1) \), and the string \( B \) after the partner is also a valid string of length \( 2(n - i - 1) \). Conversely, if you give me any two strings of valid parentheses of combined length \( 2(n - 1) \), I can recreate one of length \( 2(n) \) by \( (A)B \). It follows that the number of strings of valid parentheses where the first parenthesis pair off with the \( i \)th right parenthesis equals \( p(i - 1) \times p(n - i) \). If we sum over all \( i \), we get the recurrence.

This solves to

\[
p(n) = \frac{1}{n+1} \binom{2n}{n}
\]

While we don’t show how to find the solution, you can verify the solution yourself.

Exercises

11.1. Prove that if an \( n \)-term arithmetic progression has first term \( A \) and last-term \( L \), then its sum is \( n(A + L)/2 \).

11.2. Consider a board like a checkerboard that is partitioned into squares. Define a tromino tiling of a board to mean covering the board completely with nonoverlapping trominoes, where each tromino covers three squares in a row (horizontally or vertically). Let \( t(n) \) be the number of tromino tilings of the \( 3 \times n \) board. Give a recurrence formula for \( t(n) \).

11.3. In a rabbit warren, each pair of rabbits aged two months or more produces 2 pairs per month (and never dies).

(a) Give a recurrence for the number of pairs after \( n \) months.

(b) If we start with 1 newborn pair, how many rabbits do we have after one year?

11.4. Prove by induction that the Fibonacci number \( f(4m - 1) \) is a multiple of 3 for all \( m \geq 1 \).

11.5. Let \( S(n) \) be the number of strings of length \( n \) consisting of 0s and 1s such that no two 1s are consecutive. Determine a recurrence formula for \( S(n) \).

11.6. Prove that the Fibonacci sequence obeys the following identity:

\[
f(0) + f(1) + \ldots + f(n) = f(n + 2) - 1.
\]

11.7. Prove that the Fibonacci numbers obey the following identity:

\[
\sum_{i=0}^{n} [f(i)]^2 = f(n)f(n+1) \quad \text{for } n \geq 0.
\]
11.8. Prove that every positive integer can be written as a sum of some distinct Fibonacci numbers with the added restriction that no two of the Fibonacci numbers used are consecutive. For example, 28 = f(7) + f(4) + f(2).

11.9. (a) Show that any two consecutive Fibonacci numbers \( f(n-1) \) and \( f(n) \) are relatively prime.
(b) Code up or apply the Extended Euclid algorithm to find \( s \) and \( t \) such that \( sf(n) + tf(n - 1) = 1 \). Discuss your results, conjecture a pattern, and try to prove your conjecture.

11.10. Verify that the formula for \( p(n) \), the number of valid strings of \( 2n \) parentheses, is correct by showing that it satisfies the recurrence.

11.11. Wayne has a membership at the FoolTiltPoker website that allows him to play any days that he likes, except that he can never play on three consecutive days. Let \( p(n) \) be the number of possible subsets of days in an \( n \)-day period when Wayne can play.
(a) Explain why \( p(3) = 7 \).
(b) Calculate \( p(4) \).
(c) Give a recurrence for \( p(n) \).