12 Solving Recurrences

12.1 Iterating the Recurrence

We can solve some recurrences by iterating them. This means repeatedly using the recurrence relation to re-write the RHS. (Actually, we can often get some information about them this way.)

For example: For our money from Example 11.3:

\[ M(n) = (1 + p/100)M(n - 1) = (1 + p/100)^2 M(n - 2) = \ldots = (1 + p/100)^n A. \]

And for our pairs from Example 11.2:

\[ P(n) = (n - 1) + (n - 2) + \ldots + 1 + 0 = n(n - 1)/2 \]

where the last part uses the formula for the sum of an arithmetic progression.

Here is a harder example of solving a recurrence using iteration.

\[ \text{Example 12.1. Solve} \]

\[ T(n) = 4T(n - 1) + 2^n \quad \text{with } T(0) = 6. \]

Iterating the recurrence:

\[
T(n) = 2^n + 4T(n - 1) \\
= 2^n + 4(2^{n-1} + 4T(n - 2)) \\
= 2^n + 2^{n+1} + 4^2T(n - 2) \\
= 2^n + 2^{n+1} + 2^{n+2} + 4^3T(n - 3) \\
= \ldots \\
= (2^n + 2^{n+1} + \ldots + 2^{2n-1}) + 4^n T(0) \\
= (2^{2n} - 2^n) + 6 \cdot 4^n \\
= 7 \cdot 4^n - 2^n
\]
12.2 Characteristic Equations

The recurrence relation for the Fibonacci numbers is a second-order recurrence, meaning it involves the previous two values. It is also linear homogeneous, meaning that every term is a constant multiplied by a sequence value. In general, one can write this as:

\[ g(n) = ag(n - 1) + bg(n - 2). \]

Now, it turns out that \( g(n) = r^n \)—where \( r \) is some fixed real number—is a solution to this recurrence under certain circumstances. What are those circumstances? Well, a trivial case is \( r = 0 \); but let’s assume \( r \neq 0 \). One can plug this alleged solution into both sides and see what must happen. The LHS is \( r^n \). The RHS is \( ar^{n-1} + br^{n-2} \). If we divide through by \( r^{n-2} \) (legal since \( r \neq 0 \)), we get that \( r^2 = ar + b \). Put another way, we need \( r \) to be a root (that is, a solution) of the following equation:

\[ x^2 = ax + b. \]

This is called the characteristic equation.

**Theorem 12.1** If the characteristic equation \( x^2 = ax + b \) has two distinct real roots \( r_1 \) and \( r_2 \), then the solution of the recurrence relation \( g(n) = ag(n - 1) + bg(n - 2) \) \( (n \geq 2) \) is given by

\[ g(n) = \alpha r_1^n + \beta r_2^n, \]

where \( \alpha \) and \( \beta \) are real numbers.

**Proof.** We have just shown that each of \( g(n) = r_1^n \) and \( g(n) = r_2^n \) is a solution. The theorem claims that these two functions are, in the terms of linear algebra, a basis for the solution space: every other solution is a linear combination of these two, and every linear combination of these two is indeed a solution. We omit the proof, but you should do it if you need the exercise. \( \diamond \)

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**Example 12.2.** Solve the recurrence \( R(n) = 5R(n - 1) - 6R(n - 2) \).

Method: the above theorem applies. The characteristic equation is \( x^2 = 5x - 6 \). We first solve the quadratic: the roots are \( r_1 = 2 \), \( r_2 = 3 \). So the general formula is \( R(n) = \alpha 2^n + \beta 3^n \) for some constants \( \alpha \) and \( \beta \). The constants \( \alpha \) and \( \beta \) can be obtained by looking at the initial conditions, which are the first two values of the sequence. We get two equations in two unknowns, which we then solve.

Let’s determine the solution for the the Fibonacci numbers. The characteristic equation is \( x^2 = x+1 \). By the quadratic formula, the roots of this are \( r_1 = (1+\sqrt{5})/2 \) and \( r_2 = (1-\sqrt{5})/2 \). So the solution is \( f(n) = \alpha r_1^n + \beta r_2^n \).
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The coefficients $\alpha$ and $\beta$ are found by using the initial conditions, that is, that $f(0) = f(1) = 1$. In particular, we need that

$$f(0) = \alpha + \beta = 1 \quad \text{and} \quad f(1) = \alpha r_1 + \beta r_2 = 1.$$

And we get from algebra, that $\alpha = r_1/\sqrt{5}$ and $\beta = -r_2/\sqrt{5}$. This means that

$$f(n) = \frac{1}{\sqrt{5}} \left( \left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1} \right).$$

But things are actually simpler than they look: as $n \to \infty$ the second term tends to zero, so actually

$$f(n) \approx \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^{n+1}.$$

(Indeed, it can be shown that $f(n)$ is the nearest integer to this quantity.)

One can also establish a similar result to the above theorem for the case where the two roots are not real, or for the case where there is a repeated root. The ideas for the latter are discussed in the exercises.

**Exercises**

12.1. Solve the recurrence $z(n) = 2z(n-1) + 4^n$ with $z(0) = 1$ by iterating the recurrence.

12.2. Solve the recurrence $s(n) = 3s(n-1) + 1$ with $s(0) = 1$ by iterating the recurrence.

12.3. The Lucas numbers $l(n)$ are like the Fibonacci numbers, except that they start differently: 2, 1, 3, 4, 7, 11, 18 . . . State and prove a formula for the sum of the first $n$ Lucas numbers.

12.4. Determine a formula for the Lucas numbers $l(n)$.

12.5. Give the general solution of the recurrence $g(n) = 2g(n-1) + 3g(n-2)$ by using the characteristic equation.

12.6. Solve the recurrence $h(n) = 6h(n-1) - 8h(n-2)$, with $h(1) = 4$ and $h(2) = 16$, by using the characteristic equation.

12.7. Consider the recurrence $h(n) = 4h(n-1) + 4h(n-2)$ ($n \geq 2$).

(a) Show that the characteristic equation has only one root.

(b) Show that both $h(n) = 2^n$ and $h(n) = n2^n$ are solutions to the recurrence.

(c) Suppose $h(0) = 1$ and $h(1) = 2$. Solve the recurrence.