3 Properties of Binomial Coefficients

3.1 Double Counting and Combinatorial Proofs

We consider next a famous fact about binomial coefficients.

Theorem 3.1 For $n \geq 0$,
\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.
\]

One way to show that a formula like this is true, is to produce a counting problem $Y$ such that if you look at it one way the answer to $Y$ is the left-hand-side, and if you look at another way the answer to $Y$ is the right-hand-side. This explains why something is true. We call it a combinatorial proof.

Proof. We give a combinatorial proof. Let $X$ be an $n$-element set, and let $Y$ be the set of subsets of $X$. In Example 1.4 we observed that $|Y| = 2^n$.

On the other hand, let us count $Y$ by considering the sizes of each subset. Then, by Lemma 2.1 there are $\binom{n}{k}$ of size $k$, and so, if we sum this quantity from $k = 0$ to $k = n$, we get $|Y|$. Thus the two sides of the above equation are in fact equal. ♦

If we use what is called sigma-notation, then the above equation can be rewritten as
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n.
\]

In the expression $\sum_{k=0}^{n}$, it means to loop through all values from $k = 0$ to $k = n$, evaluate the formula, and add up all the results.

3.2 A Recurrence for Binomial Coefficients

Here is the famous recursive formula for binomial coefficients.

Lemma 3.2 For $1 \leq k < n$,
\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]
This equation can be proven by replacing each binomial coefficient by its ratio of factorials and checking that we get the same on both sides. (Do it!) But we give a combinatorial proof.

**Proof.** Let $Y$ be the unordered subsets of size $k$. In the above equation, the LHS (left-hand side) by definition counts $Y$. So what about the RHS (right-hand side)?

Let $a$ be the first element of the universe. A subset either contains $a$ or it doesn’t. If the subset contains $a$, then what remains is a subset of size $k - 1$ from the remaining universe of size $n - 1$. If the subset does not contain $a$, then it is a subset of size $k$ from the remaining universe of size $n - 1$. So by the sum rule, the RHS also counts $Y$ (the unordered subsets of size $k$): the first binomial coefficient counts those with $a$ and the second binomial coefficient counts those without. 

Binomial coefficients can be arranged in what is called **Pascal’s triangle** (even though multiple cultures investigated it long before Pascal). Pascal’s triangle has the rule that each entry is the sum of the two entries immediately above it, and so the $n^{th}$ row from the top is the binomial coefficients \( \binom{n}{k} \). Many thousands of pages have been written about the properties of binomial coefficients and their kin.

For example, the remainders when binomial coefficients are divided by a prime provide interesting patterns. Here is the start of Pascal’s triangle with the odd binomial coefficients shaded.
3.3 The Binomial Theorem

The result in Theorem 3.1 is generalized in the famous Binomial Theorem. (It’s a generalization, because if we plug \( x = y = 1 \) into the Binomial Theorem, we get the previous result.)

**Theorem 3.3 (Binomial Theorem)**

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

**Proof.** Let’s start by showing the idea in a specific case. Consider \( n = 3 \). Then the LHS product is \((x + y)(x + y)(x + y)\). If we multiply this out, but do not use the commutative law for multiplication, we get \(xxx + xxy + xyx + yxx + yxy + yyx + yyy\). Now, to get the coefficient of \(x^2y\) say, we group together \(xxy, xyx, \) and \(yxx\). That is, the coefficient of \(x^2y\) is the number of ways of creating a “word” using exactly \(x, x, \) and \(y\). To count such, we choose the positions for the \(y\)’s: this is a subset of size \(k\).

The real proof is exactly the above idea but with notation. The total number of \(x^{n-k}y^k\) in \((x + y)^n\) is equal to the number of ways of placing the \(k\) \(y\)’s in a word together with \(n - k\) \(x\)’s; this is the binomial coefficient \((\frac{n}{k})\).

\[\diamondsuit\]

**Example 3.1.** What is the coefficient of \(x^2y^5\) if we multiply out \((x + y)^7\)?

By the Binomial Theorem, it is \((\binom{7}{5})\) (or \((\binom{7}{2})\) if you prefer).

**For you to do!**

1. Take several deep breaths.

**Exercises**

3.1. Provide a combinatorial proof of the identity:

\[
n \binom{n - 1}{2} = \binom{n}{2} (n - 2)
\]

(Hint: Consider a three-person subcommittee with a leader.)

3.2. (a) Show that

\[
\binom{n}{k} = \frac{n}{k} \binom{n - 1}{k - 1}
\]

provided \(k\) is positive.
(b) Give a combinatorial proof of this.

3.3. Consider the equation
\[
\binom{n}{200} \binom{200}{20} = \binom{n}{20} \binom{n-20}{180}
\]
(a) Use algebra and the formula for binomial coefficients to prove this equation.
(b) Provide a combinatorial proof of this equation. (Hint: consider choosing a committee and a subcommittee.)

3.4. Show that if \( p \) is a prime number, then \( \binom{p}{i} \) is a multiple of \( p \) for all \( i \) from 1 up to \( p-1 \).

3.5. Consider the following identity:
\[
\sum_{0 \leq i \leq n/2} \binom{n}{2i} = \sum_{0 \leq i < n/2} \binom{n}{2i+1}.
\]
For example, when \( n = 3 \) it claims that \( \binom{3}{0} + \binom{3}{2} = \binom{3}{1} + \binom{3}{3} \); this is true since both sides equal 4.
(a) Verify this identity for \( n = 4 \) and \( n = 5 \).
(b) Deduce this identity from the Binomial Theorem (by plugging in suitable value of \( x \) and \( y \)).
(c) Give a combinatorial proof of the identity.

3.6. Consider the following identity:
\[
\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.
\]
For example, when \( n = 2 \) it claims that \( \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = \binom{4}{2} \); this is true since both sides equal 6.
(a) Verify this identity for \( n = 3 \) and \( n = 4 \).
(b) Give a combinatorial proof of the identity. (Hint: consider a \( 2n \)-element set with half its elements colored red.)

3.7. Just like \( \sum \) for addition, there is \( \prod \) for multiplication. Show that
\[
\binom{n}{k} = \prod_{i=1}^{k} \frac{n-k+i}{i}
\]

3.8. (a) Using some mathematics software or a calculator, calculate \( \binom{50}{25} \).
(b) In Java (and usually in C) an int variable has a maximum value of $2^{31}$. Explain why we cannot use int’s to calculate $50!$.

(c) Write code using the recursive formula from Lemma 3.2 to calculate $\binom{50}{25}$. (Note that you will need to stop the recursion under certain circumstances.)

(d) Write code using the formula from Exercise 3.7 to calculate $\binom{50}{25}$.

(e) Comment on the efficiency of your code.

3.9. Prove that $\binom{2000}{1000}$ is even.

3.10. Suggest and prove a generalization of the Binomial Theorem of the form $(x + y + z)^n = \sum \ldots$. 