In this section we consider (and count) mathematical objects called functions, partitions, relations, and equivalence relations.

4.1 Functions

You have seen functions before. A function has a domain and a codomain. The function maps each element in the domain to an element in the codomain; that is, given any element of the domain, the function evaluates to a specific value in the codomain. The range of a function is the set of elements in the codomain which really do have something mapping to them.

Example 4.1.

Suppose that \( f(x) = x^2 - 4 \) with domain and codomain all real numbers. Then the range is all real numbers at least \(-4\). This is equivalent to asking for which \( y \) does there exist an \( x \) such that \( y = x^2 - 4 \).

Note that the domain must be given as part of the definition of the function; so should the codomain.

There are three special types of function:

- A function is said to be one-to-one if every element in the range is mapped to by a unique element in the domain.

- A function is said to be onto if every element in the codomain is mapped to; that is, the codomain and the range are equal.

- A function is said to be a bijection if it is both one-to-one and onto.

In the above example, the function \( f \) is not one-to-one; for example, \( f(3) = f(-3) \). The function is also not onto; for example, there is no \( x \) such that \( f(x) = -7 \). The key point about a one-to-one function is that it is reversible, in that if you tell me \( f(x) \) I can work out \( x \).

Example 4.2. For each of the following functions, the domain and the codomain are the set of all integers. Determine which of the functions are one-to-one and
which are onto.

(a) \( f(x) = x + 1 \)
(b) \( f(x) = 2x \)
(c) \( f(x) = \frac{x}{2} \) if \( x \) is even and \( \frac{x+1}{2} \) if \( x \) is odd.

(a) This is a bijection.

(b) This is one-to-one but not onto. (The range is all even integers.)

(c) This is onto but not one-to-one. (For example, \( f(1) = f(2) \).)

Some people (including me) like to represent a function using a collection of arrows. The domain is on the left and the codomain on the right and there is exactly one arrow leading out of each element on the left.

Example 4.3. Here is a depiction of a bijection from \( \{a, b, c\} \) to \( \{d, e, f\} \).

We observe the following elementary properties of functions, whose proof we leave as an exercise. (Actually, this lemma requires a bit of thought—for example, I got range and codomain mixed up the first time I wrote it down... sigh.)

Lemma 4.1 Let \( f \) be a function with a finite domain and codomain.
(a) The range is at most the size of the domain.
(b) If \( f \) is one-to-one, then the codomain is at least as large as the domain.
(c) If \( f \) is onto, then the codomain is at most as large as the domain.
(d) If \( f \) is a bijection, then the domain and range are the same size.
(e) If the domain and range are the same size, then \( f \) is onto if and only if it one-to-one if and only if it is a bijection.

Example 4.4. Let \( A = B = \{0, 1\} \).
(a) How many functions are there from \( A \) to \( B \)?
(b) How many onto functions are there from \( A \) to \( B \)?
(c) How many one-to-one functions are there from \( A \) to \( B \)?
(d) How many bijections are there from \( A \) to \( B \)?
(a) To specify each function, we must specify where each member of A gets mapped to. That is, we choose what 0 gets mapped to and what 1 gets mapped to. We have two choices for each, so the answer is $2^2 = 4$.

(b,c,d) By Lemma 4.1e, the answers to these three parts are the same. A bijection means we pair off elements of A with elements of B. There are only two possibilities: the function that maps $0 \rightarrow 0$ and $1 \rightarrow 1$, and the function that maps $0 \rightarrow 1$ and $1 \rightarrow 0$.

▶ For you to do! ◀

Let $S = \{a, b, c, d, e\}$.

1. How many functions from S to S are there?
2. How many one-to-one functions from S to S are there?

### 4.2 Partitions

A partition of a set is writing it as the disjoint union of nonempty blocks. (Recall that disjoint means non-overlapping.) For example, $\\{\{1\}, \{3, 5\}, \{2, 4\}\}$ is a partition of the set $X = \{1, 2, 3, 4, 5\}$. Note that the order of the blocks does not matter, and neither does the order of the elements within a block.

Example 4.5. Determine all partitions of the set $\{a, b, c\}$.

There are 5 partitions. There is 1 partition into one block. There is 1 partition into three blocks (where every element is in a block by themselves). There are 3 partitions into two blocks: based on which of the elements is in a block by itself. We might write these partitions as:

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abc
|  a | bc | b | ac | c | ab |
\|  a | b | c |
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### 4.3 Relations

In English we often say something is “related to” or “similar to” or “connected to” something else. This could be because they share genes, or one thing causes the other thing, or because they are different colors. Mathematics tries to capture this notion with what it calls a “relation”. To specify a relation, we can give a rule which explains when two things are related, for example, when they are different colors. More generally, we can specify a function
by listing all the pairs of related elements. Examples of relations include “same color as”, “is a subset of”, and “is a neighbor of”.

Mathematically, a relation on a universe $X$ is a set of ordered pairs on $X$. If $R$ stands for the relation, then we will write $xRy$ to mean that $x$ is related to $y$ in the relation $R$. For example, if $R$ was the “equality relation”, we would write that $x = y$. Though we will do it in the following example, it is usually impossible to write out all the ordered pairs, since there are often infinitely many of them.

Example 4.6. Assume the universe is $X = \{0, 1, 2, 3\}$. What is the usual name for the following relations?

(a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
(b) $\{(0, 1), (1, 2), (2, 3), (0, 2), (1, 3), (0, 3)\}$
(c) $\{(0, 0), (1, 1), (2, 2), (3, 3), (1, 3), (0, 2), (2, 0), (3, 1)\}$

(a) Equal to
(b) Less than
(c) Has the same parity as (same remainder when divided by 2).

4.4 Equivalence Relations

We are interested in relations that have three specific properties:

- A relation $R$ is reflexive if $xRx$ for all $x$ (that is, everything is defined to be related to itself).
- A relation $R$ is symmetric if $xRy$ always implies $yRx$; and
- A relation is transitive if $xRy$ and $yRz$ being true always implies $xRz$.

For example, the relation “less than” is transitive: if $a < b$ and $b < c$ then it necessarily follows that $a < c$. And, the relation “is a neighbor of” is symmetric: if I’m your neighbor then necessarily you’re my neighbor.

An equivalence relation is one that is reflexive, symmetric, and transitive.

Example 4.7. Consider all the people in the world. We can define two people to be related if they have the same name. Show that this relation, call it $S$, is an equivalence relation.

There are three properties to check. I have the same name as myself; so $S$ has the reflexive property. If I have the same name as you, then you have the same name as me; so $S$ has
the symmetric property. If person $X$ has the same name as person $Y$ and person $Y$ has the same name as person $Z$, then certainly persons $X$ and $Z$ have the same name; so $S$ has the transitive property. That is, the three properties of an equivalence relation are satisfied.

In an equivalence relation, the **equivalence class** of element $x$ is the set of all elements related to it (note that $x$ is in its own equivalence class, by the reflexive property). That is, an equivalence class is a set of elements that are considered to be similar or equivalent.

**Example 4.8.** Let $\mathbb{N}$ be the set of all nonnegative integers. Define the relation $M$ so that $xM y$ if and only if $x$ and $y$ have the same units digit. Show that $M$ is an equivalence relation and determine the equivalence classes.

This is an equivalence relation. For example, to check that it is transitive, we note that if $x$ and $y$ end in the same digit, and $y$ and $z$ end in the same digit, then it must be the case that $x$ and $z$ end in the same digit.

There are 10 equivalence classes—one for all numbers ending with a 0, one for all numbers ending with a 1, and so on.

**Theorem 4.2** For any equivalence relation $R$, any two equivalence classes are either disjoint or equal.

**Proof.** Let $R$ denote the equivalence relation, and let $E_x$ and $E_y$ denote the equivalence class containing $x$ and $y$ respectively. Assume that $E_x$ and $E_y$ are not disjoint. Then that means there is some common element, call it $z$.

Now, let $a$ be some element of $E_x$. By definition, this means that $aRx$. Further, we have that $xRz$ (since $z$ is in $E_x$) and that $zRy$ (since $z$ is in $E_y$). So it follows that $aRy$ (by the transitive property). That is, $a \in E_y$. And the converse holds: if $b \in E_y$, then $b \in E_x$.

That is, we have shown that every element of $E_x$ is also an element of $E_y$ and vice versa. This means that $E_x = E_y$. ♦

The theorem means that if you tell me the equivalence classes, I can work out the equivalence relation. In particular, the theorem means that the equivalence classes form a partition of $X$. But the connection goes the other way too: every partition gives rise to an equivalence relation. (Think about why...)

**Lemma 4.3** If $X$ is some universe, there is a bijection between the set of equivalence relations on $X$ and the set of partitions of $X$. 
Equivalence relations are useful in counting. Indeed, we already implicitly did this, when we said two things were to be considered the same even though we counted them twice. Recall that we proved that the number of $k$-element subsets of an $n$-element set is $\binom{n}{k}$, by counting $k$-element sequences and arguing that each $k$-element subset arose from $k!$ such sequences.

In general, if we count some set $X$ by counting some process that generates elements of $X$, then we have to divide by the number of ways each element of $X$ is produced. This can stated as the Quotient Principle:

**Lemma 4.4** If we partition a universe of size $p$ into $q$ blocks of size $r$, then $q = p/r$.

▶ For you to do! ◀

Let $Z$ be the set of all integers (positive and negative). In each case, determine whether the relation on $Z$ is an equivalence relation or not, and justify your answer.

3. $N$ is the “nothing” relation. That is, no element is related to any other element, not even itself.

4. $A$ is the “absolute value” relation. That is, two elements are related if their absolute value is the same.

5. $B$ is the “bigger than”. That is, $xB$ if and only if $x > y$.

**Exercises**

4.1. Suppose $|A| = |B| = 100$. How many functions are there from $A$ to $B$? How many of these functions are bijections?

4.2. Convince your grandmother that Lemma 4.1 is true.

4.3. Consider the function $f(x) = x^2$.

   (a) Give an example of domain and codomain such that the function $f$ is onto but not 1–1.

   (b) Give an example of domain and codomain such that the function $f$ is 1–1 but not onto.

   (c) Give an example of domain and codomain such that the function $f$ is a bijection.

4.4. Let $A = \{a, b\}$ and $B = \{c, d, e\}$.

   (a) How many functions are there from $A$ to $B$?

   (b) How many onto functions are there from $A$ to $B$?
(c) How many one-to-one functions are there from $A$ to $B$?
(d) How many bijections are there from $A$ to $B$?

4.5. Let $Y = \{t, u, v, w\}$ and $Z = \{x, y, z\}$.
(a) How many functions are there from $Y$ to $Z$?
(b) How many onto functions are there from $Y$ to $Z$?
(c) How many one-to-one functions are there from $Y$ to $Z$?
(d) How many bijections are there from $Y$ to $Z$?

4.6. List all partitions of the set $\{a, b, c, d\}$. (Hint: there are 15.)

4.7. How many partitions are there of a 5-element set?

4.8. In how many ways can an 100-element set be partitioned into
(a) 101 blocks?
(b) 100 blocks?
(c) 99 blocks?
(d) 98 blocks?

4.9. Let $S(n, k)$ denote the number of partitions of an $n$-element set into a partition with $k$ blocks.
(a) Explain why $S(n, 1) = 1$, $S(n, n) = 1$, and $S(n, k) = 0$ if $k > n$.
(b) Explain why $S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$.
(c) Use this to calculate the number of partitions of a 6-element set.

4.10. Consider the set $S = \{a, b, c, d, e\}$ and the partition $P = \{\{a\}, \{b, c\}, \{d, e\}\}$. Write down the ordered pairs of the equivalence relation on $S$ whose equivalence classes are given by $P$.

4.11. Let $<$ be the “less-than” relation with universe the positive integers;
let $D$ be the relation with universe all sets, such that two sets are related if they are disjoint
let $\approx$ be the relation with universe all real numbers, such that $x$ and $y$ are related if $|x - y| < 0.01$.

Complete the following table (with “yes” and “no”):
4.12. Let the universe be $\mathbb{N}$ the set of all nonnegative integers. In each of the following, determine whether the relation is an equivalence relation. If it is not, state one property it fails to have; if it is, state the number of equivalence classes.

(a) $E$ is the “everything” relation. That is, every number is related to every number.
(b) $N$ is the “nothing” relation. That is, no number is related to any other number, not even itself.
(c) $P$ is the “parity” relation. That is, two numbers are related if they are both even, or if they are both odd.
(d) $L$ is the “less than or equal” relation. That is, $x$ is related to $y$ if $x \leq y$.

4.13. Let the universe be all 178,691 words in the official English Scrabble dictionary. In each of the following, determine whether the relation is an equivalence relation. If it is not, state one property that the relation fails to have. If it is, state the number of equivalence classes.

(a) $S$ is the “start” relation. That is, words $x$ and $y$ are related if they have the same initial letter.
(b) $L$ is the “loner” relation. That is, every word is related to itself but not to any other other word.
(c) $N$ is the “nothing” relation. That is, no word is related to any word, not even themselves.
(d) $A$ is the “alphabetical” relation. That is, $xAy$ if $x$ occurs before $y$ in the dictionary.
(e) $O$ is the “one-letter” relation. That is, $xOy$ if $x$ and $y$ differ by exactly one letter.

4.14. Give an example of a symmetric relation that is:

(a) Reflexive and transitive
(b) Reflexive and not transitive
(c) Transitive and not reflexive

4.15. Let $A$ be the set of all positive integers and let $X = A \times A$. Define a relation $R$ on $X$ by saying that $(a,b) R (c,d)$ iff $ad = bc$. Show that $R$ is an equivalence relation. Would $R$ still be an equivalence relation if $A$ was the set of all integers?
4.16. Let $X$ be the set of all words in the English dictionary. In each case, determine whether the relation on $X$ is an equivalence relation or not. If it is an equivalence relation, determine how many equivalence classes there are. If it is not an equivalence relation, state one of the three conditions the relation does not obey.

(a) $G$ is the “geography” relation. That is, $xGy$ if word $y$ begins with the same letter that word $x$ ends with (for example, CAT is related to TIGER but not vice versa).

(b) $F$ is the “first-letter” relation. That is, $xFy$ if words $x$ and $y$ begin with the same letter.

(c) $P$ is the “contains-p” relation. That is, two words are related if either they both contain a p, or if neither contains a p.

4.17. Some books define a function as a special type of relation. Suggest how such a definition might go.

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**Solutions to Practice Exercises**

1. $5^5$ (five choices for each element of the domain).

2. $5!$ (five choices for $f(a)$, four choices for $f(b)$, and so on.)

3. Not an equivalence relation. For example, not reflexive.

4. Is an equivalence relation.

5. Not an equivalence relation. For example, not symmetric.