

A very brief introduction to Graph Theory (minus the proofs)

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1 The Basics	2
1.1 Graphs	2
1.2 Degrees and Trees	3
1.3 Example Graphs	3
1.4 Bipartite Graphs	4
2 Paths, Cycles, and Connectivity	5
2.1 Eulerian Graphs	5
2.2 Cycles and Hamiltonian Graphs	5
2.3 Connectivity	6
2.4 Directed Graphs	7
2.5 Distance and Searches	7
3 Colorings, Matchings, and Graph Families	9
3.1 Vertex Colorings	9
3.2 Edge Colorings	10
3.3 Matchings	10
3.4 Planar Graphs	11
3.5 Graph Operations	11

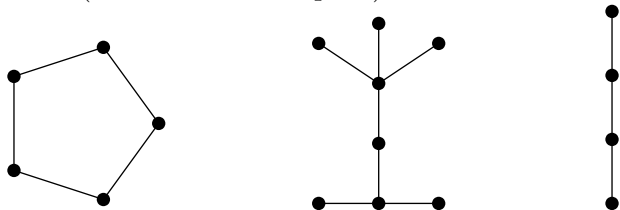
1 The Basics

1.1 Graphs

A (simple) **graph** is a collection of vertices and edges such that each edge joins two vertices. People sometimes allow **multiple edges** between vertices (for example, to represent double-bonds) or **loops** (edges both of whose ends are the same vertex), but we exclude those here—that is the meaning of “simple” in simple graph.

A **walk** is a sequence of vertices such that consecutive vertices are joined by an edge. The **length** of a walk is the number of edges on it. A **path** is a walk without repeated vertices. A **cycle** is a walk of at least three edges without repeated vertices except that the first and last vertex are the same. The terms **path** and **cycle** also refer to the specific graphs that have that structure.

Two vertices are **connected** if there is a walk between them. Being-connected is an equivalence relation; the equivalence classes form the **components** of the graph. A graph is **connected** if there is only one component. A **tree** is a graph that is connected and contains no cycle. Here is a graph with three components: a cycle and two trees (one of which is a path).



A **subgraph** of a graph G is a graph that contains some of the edges and some of the vertices of the graph G . A subgraph is a **spanning** subgraph if it contains all the vertices of the original graph. An **induced subgraph** is one which contains all possible edges for those vertices.

Two graphs are **isomorphic** if there is a bijection between the vertex sets that preserves the edges.

1.2 Degrees and Trees

The **degree** of a vertex is the number of edges coming out of it. A graph is **regular** if all vertices have the same degree. The following is sometimes called the “First Theorem of Graph Theory”:

Lemma 1.1 *Suppose the graph has n vertices and a edges. Suppose the degrees of the graph are d_1, \dots, d_n . Then $\sum_{i=1}^n d_i = 2a$.*

This is a double-counting argument: the two sides of the equation count the same quantity, namely “ends of edges”. As a consequence we get that in any graph, the number of vertices of odd degree is even.

A **leaf** is a vertex of degree 1.

Lemma 1.2 (a) *Between any two vertices in a tree there is a unique path.*

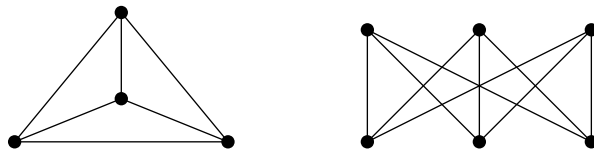
(b) *Removing any edge disconnects the tree.*

(c) *If a tree has n vertices, then it has $n - 1$ edges.*

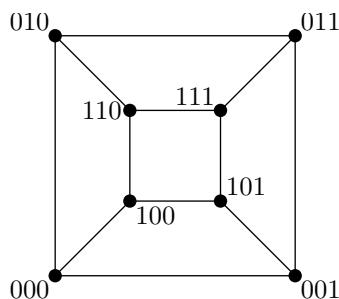
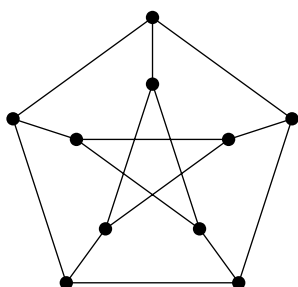
(d) *If a tree has at least two vertices, then it has at least two leaves.*

1.3 Example Graphs

As a graph, we use P_n to denote the **path** on n vertices and C_n to denote the **cycle** on n vertices. The **complete graph** K_n on n vertices is the graph where every pair of vertices are joined by an edge. The **complete bipartite graph** $K_{r,s}$, for positive integers r and s , has $r + s$ vertices, split into two groups with r vertices on one side and s vertices on the other, and all edges are present between the two sides. A **star** is $K_{1,n-1}$. More generally, a **complete multipartite graph** is a simple graph where the vertex set can be partitioned into subsets such that two vertices are adjacent if and only if they are in different subsets. Here are K_4 and $K_{3,3}$.



The **Petersen graph** has 10 vertices, is 3-regular, and the shortest cycle has length 5. The **hypercube** Q_q of dimension q has 2^q vertices. Each vertex corresponds to a bit string (meaning 0's and 1's) of length q . Two vertices are joined by an edge if their corresponding strings differ in **exactly** one bit. Each vertex therefore has degree q . We can also define Q_q recursively: To form Q_q , take two copies of Q_{q-1} and join each pair of corresponding vertices by an edge. Here is the Petersen graph and Q_3 .



1.4 Bipartite Graphs

We saw already the complete bipartite graph. In general, a graph is **bipartite** if one can partition the vertices into two sets, such that each edge has an end in each set. A tree is bipartite. To color a tree: pick a root, color the root say red, color its children say blue, color their children (the root's grandchildren) red, and so on alternating; every edge has one red end and one blue end.

Theorem 1.3 *A connected graph is bipartite if and only if every cycle has even length.*

The hypercube is bipartite.

2 Paths, Cycles, and Connectivity

2.1 Eulerian Graphs

For a famous example of a problem, consider the problem of drawing a picture without lifting your pen and without going over the same line more than once. An **Euler tour** is a walk that goes along every edge exactly once, and ends up where one started. This is like the continuous pen drawing, except with the added requirement that one ends at the same place one begins.

Theorem 2.1 *A connected graph has an Euler tour if and only if every vertex has even degree.*

Note that there are two things to prove: that if the graph has an Euler tour, then every vertex has even degree; and if every vertex has even degree, then the graph has an Euler tour.

2.2 Cycles and Hamiltonian Graphs

A **Hamilton cycle** is a cycle that visits every vertex exactly once. That is, a Hamilton cycle is a spanning cycle. Similarly, a **Hamilton path** is a path that visits every vertex exactly once. This idea sounds similar to Euler, but not really. No simple characterization of when a graph has a Hamilton cycle is known. Indeed, it is strongly believed that such a characterization does not exist, since it has been shown to be **NP-complete**.

The complete graph K_n has a Hamilton cycle for $n \geq 3$; the complete bipartite graph $K_{r,s}$ has a Hamilton cycle if and only if $r = s \geq 2$. The hypercube has a Hamilton cycle; indeed, Hamilton cycles in the hypercube are called **Gray codes**, and are important in communication. Here is one sufficient condition and one necessary condition.

Theorem 2.2 *(a) Let G be a graph with n vertices. If every vertex has degree at least $n/2$, then G has a Hamilton cycle.*

(b) If graph G has a Hamilton cycle, then for every set S of vertices, the number of components of $G - S$ is at most $|S|$.

The converse of this theorem is false.

The **girth** of a graph is the length of the shortest cycle. The **circumference** is the length of the longest cycle. A graph is **triangle-free** if its girth is at least 4.

Theorem 2.3 For a triangle-free simple graph with n vertices, the maximum number of edges is $\lfloor n^2/4 \rfloor$.

In general, the maximum number of edges in a graph which has no K_{k+1} is achieved by taking a complete k -partite graph with as equal parts as possible.

2.3 Connectivity

The **connectivity** $\kappa(G)$ of graph G is the minimum number of vertices whose removal disconnects the graph, except that we define $\kappa(K_n) = n - 1$. The **edge-connectivity** $\kappa'(G)$ is the smallest cardinality of a disconnecting set of edges, except that we define $\kappa'(K_1) = 0$. For example, the hypercube Q_k has connectivity k . A graph with $\kappa(G) \geq k$ is called **k -connected**.

A simple observation is that if a graph is k -connected, then every vertex has degree at least k (since the removal of a vertex's neighbors disconnects the graph). Indeed we have:

Lemma 2.4 For any graph, $\kappa \leq \kappa' \leq \delta$ where δ denotes the minimum degree.

A **cut-vertex** is a vertex whose removal disconnects its component. A **cut-edge** (or bridge) is similar; an edge is a cut-edge if and only if it belongs to no cycle. A **block** is a maximal connected subgraph that has no cut-vertex.

Theorem 2.5 [Whitney] A graph is k -connected if and only if there are k internally disjoint (meaning no vertex in more than one path) paths between every pair of vertices.

A special case is that a graph is 2-connected if and only if for every pair of vertices there is a cycle containing them. Similar results hold for edge-connectivity.

2.4 Directed Graphs

A **directed graph** has directed **edges/arcs**; each arc goes from **in-neighbor** to **out-neighbor**. The **in-** or **out-degree** of a vertex is the numbers of edges going in or going out of the vertex. A **source** is a vertex with no in-arcs, and a **sink** is one with no out-arcs. An **orientation** of an undirected graph is obtained by orienting every edge.

A path in a digraph must respect the arrows. A directed graph is **strongly connected** if there is a path from every vertex to every other vertex. A **DAG**, directed acyclic graph, is a directed graph without directed cycles.

A **tournament** is an oriented K_n . Every tournament has a **king**: a vertex which can reach all other vertices with a path of length at most 2.

Theorem 2.6 [Robbins] *A graph has an orientation that is strongly connected if and only if it is 2-edge-connected.*

2.5 Distance and Searches

A **rooted tree** is a tree with one vertex designated the **root**. Rooted trees are normally drawn with the root at the top, and we talk of **parents** and **children** in the natural way.

A **search** of a graph is a systematic way of searching through the vertices for a specific vertex. The two standard searches are breadth-first search and depth-first search. The idea behind **breadth-first search** is to: Visit the source; then all its neighbors; then all their neighbors; and so on. The idea for **depth-first search** (DFS) is “labyrinth wandering”: keep exploring any new vertex from current vertex; when get stuck, backtrack to most recent vertex with unexplored neighbors.

The **distance** between two vertices is the minimum number of arcs/edges on path between them. For example, in a BFS in a graph, vertices are visited in order of their distance from the start. The **eccentricity** of a vertex is the maximum distance of a vertex from it. The **radius** is the minimum eccentricity. The center is those vertices of minimum eccentricity. The **diameter** is the maximum eccentricity.

A **weighted graph** has weights on the edges. In a weighted graph, the **weight** of a path is the sum of weights of arcs/edges. The **distance** between two vertices is the minimum weight of a path between them. One way to determine the distance between two vertices is to use Dijkstra's algorithm.

3 Colorings, Matchings, and Graph Families

3.1 Vertex Colorings

A **coloring** of a graph means assigning colors to each vertex such that no edge joins two vertices of the same color. A **k -coloring** means a coloring that uses (at most) k colors. A graph having a 2-coloring is the same thing as being bipartite. The **chromatic number** of a graph, denoted χ , is the minimum number of colors needed for a coloring of the vertices.

Lemma 3.1 *An even cycle has $\chi = 2$.*

An odd cycle has $\chi = 3$.

The complete graph K_n has $\chi = n$.

An **independent set** is a set of vertices such that no pair is adjacent. The **independence number** is the maximum size of an independent set. A **vertex cover** is a set of vertices intersecting all edges; a set is a vertex cover if and only if its complement is an independent set. A coloring is a partition into independent sets. A **clique** is a set of vertices such that every pair is adjacent. The **clique number** is the maximum size of a clique. Let Δ be the maximum degree of a vertex in a graph.

Lemma 3.2 *If G is a graph with clique number ω , chromatic number χ , and maximum degree Δ , then $\omega \leq \chi \leq \Delta + 1$.*

The upper bound proof is to use a greedy algorithm. Can be slightly improved:

Theorem 3.3 [Brooks] *If G is a connected graph that is not complete nor an odd cycle, then $\chi \leq \Delta$.*

It is to be noted that, while testing for bipartiteness is easy, testing whether a graph has a 3-coloring appears to be much, much harder, since it is NP-complete.

3.2 Edge Colorings

The **edge-chromatic number** (or chromatic index) χ' of a graph is the minimum number of colors needed to color the edges such that no two edges that share a vertex have the same color. For example, $\chi'(K_n)$ is $n - 1$ if n is even and n otherwise.

It is immediate that $\chi'(G) \geq \Delta(G)$. König showed that if a graph is bipartite then $\chi'(G) = \Delta(G)$. Vizing and Gupta both showed that for any simple graph $\chi'(G) \leq \Delta + 1$.

3.3 Matchings

A **matching** is a set of edges no two of which touch. A **perfect matching** involves all the vertices. A vertex is **saturated** if it is the end of an edge in the matching. Given matching M , an **M -augmenting path** is a nontrivial path where (i) the edges alternate between in and out of M , and (ii) it starts and ends at an M -unsaturated vertex.

Theorem 3.4 [Berge] *A matching M is maximum if and only if there is no M -augmenting path.*

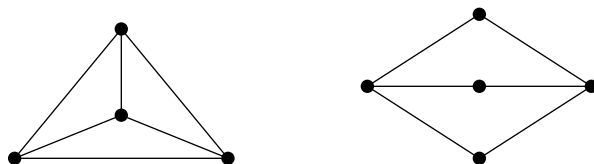
To prove existence of augmenting path if M not maximum, consider the union of M and the actual maximum matching. Berge's theorem gives an algorithm to find a maximum matching in a bipartite graph.

Theorem 3.5 [Hall] *In a bipartite graph with one side X : there is a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$, where $N(S)$ denote the set of all vertices that have at least one neighbor in S .*

Corollary: a regular bipartite graph can be decomposed into perfect matchings. Petersen's theorem says that any cubic (3-regular) graph without a cut-edge has a perfect matching.

3.4 Planar Graphs

A **plane graph** is a graph drawn in the plane such that no pair of lines intersect. The graph divides the plane up into a number of regions called **faces**. A **planar graph** is one which has a plane drawing. An **outerplanar graph** is one that can be drawn in the plane with all the vertices on one face. For example, every tree is outerplanar. Here are plane drawings of K_4 and $K_{2,3}$, neither of which is outerplanar.



Theorem 3.6 (a) [Euler's formula] *For connected plane graph with n vertices, a edges, and f faces, one has $n - a + f = 2$*
(b) *For any plane graph on n vertices and a edges, $a \leq 3n - 6$.*

Consequence: the complete graph K_5 is not planar. A **subdivision** of a graph is created by adding some number of new vertices (possibly none) on each edge.

Theorem 3.7 [Kuratowski's Theorem] *A graph is planar if and only if it does not contain a subdivision of either K_5 or $K_{3,3}$.*

The most famous theorem in this area is the 4-Color Theorem. This was one of the first major theorems to make extensive use of a computer. It is due to Appel, Haken, and Koch.

Theorem 3.8 [Four-Color Theorem] *If G is a planar graph, then $\chi(G) \leq 4$.*

3.5 Graph Operations

The **union** of two graphs is constructed by unioning their vertex sets and edge sets (throwing away duplicates). A **decomposition** of a graph is a partition of

the edges (each edge is in exactly one part of the partition). The **disjoint union** of two graphs is the union assuming their vertex sets are disjoint (and the result is disconnected).

The **complement** of a simple graph has the same vertex set but the missing edges. A graph is **self-complementary** if it is isomorphic to its complement (e.g. P_4 or C_5).

The **line graph** $L(G)$ of G has as vertex set the edges of G , with two vertices of $L(G)$ adjacent iff the corresponding edges of G share an end-vertex. For example, a matching in G corresponds to an independent set in $L(G)$. A line graph has no **claw** (induced copy of $K_{1,3}$).

The **cartesian product** of graphs G and H with vertex sets $V(G)$ and $V(H)$ has as its vertex set all pairs (u, v) of $V(G) \times V(H)$ with two pairs adjacent if the same in one coordinate and adjacent in the other. For example: hypercube $Q_k = Q_{k-1} \square K_2$. Cute theorem: $\chi(G \square H) = \max(\chi(G), \chi(H))$.