Vectors and Linear Combinations
Defn. A matrix with one column is called a (column) vector.

We use bold letters for vector variables, such as \( x \) and \( v \).

We sometimes write the column vector \[
\begin{bmatrix}
3 \\
5
\end{bmatrix}
\] as \((3, 5)\).
Vector Operations

Vector **addition** is performed by adding the corresponding entries. **Scalar multiplication** is performed by scaling each entry. That is,

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} + \begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} = \begin{bmatrix}
  u_1 + v_1 \\
  u_2 + v_2
\end{bmatrix} \quad \text{and} \quad c \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  cu_1 \\
  cu_2
\end{bmatrix}
\]

For example

\[
x \begin{bmatrix}
  2 \\
  4
\end{bmatrix} + y \begin{bmatrix}
  -1 \\
  7
\end{bmatrix} = \begin{bmatrix}
  2x - y \\
  4x + 7y
\end{bmatrix}
\]
**Defn.** We use $\mathbb{R}^d$ for the set of all $d$-entry vectors whose entries are real numbers.

One can associate vector in $\mathbb{R}^d$ with the corresponding point. For example, $\mathbb{R}^2$ is the 2-dimensional plane. And vector addition can be illustrated with a parallelogram:
Linear Combinations

**Defn.** A *linear combination* of vectors is formed by summing some multiple of each vector. The multipliers are called the *weights.*
Defn. The span of a collection of vectors is the set of all possible linear combinations. If \( S \) is a set, we will denote its span by \( \text{Span } S \). 

For example, the span of a single (nonzero) vector is a line.

The span of two vectors is (usually) a plane.
**Matrix-Vector Multiplication**

**Defn.** If $A$ is an $m \times n$ matrix and $x$ is in $\mathbb{R}^n$, then the **matrix-vector product** $Ax$ is the linear combination of the columns of $A$ specified by $x$.

That is, if $A = [a_1, \ldots, a_n]$ (meaning its columns are vectors $a_1, \ldots, a_n$), and $x = (x_1, \ldots, x_n)$ then

$$Ax = x_1a_1 + x_2a_2 + \ldots + x_na_n$$
Example of Matrix-Vector Multiplication

For example,

\[
\begin{bmatrix}
2 & -1 \\
4 & 7
\end{bmatrix}
\begin{bmatrix}
3 \\
5
\end{bmatrix}
= 3
\begin{bmatrix}
2 \\
4
\end{bmatrix}
+ 5
\begin{bmatrix}
-1 \\
7
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
47
\end{bmatrix}
\]
Summary

A vector is a matrix with one column. We use bold letters for vector variables. $\mathbb{R}^d$ is all $d$-entry vectors with real entries. Vector addition adds corresponding entries; scalar multiplication scales each entry.

A linear combination of vectors is any sum of some multiple of each vector. Their span is the set of all possible linear combinations. The product of matrix $A$ with vector $x$ is the linear combination of columns of $A$ given by $x$. 