Matrix Inverses
The Inverse of a Matrix

**Defn.** The *inverse* of a square matrix $A$, denoted $A^{-1}$, is the matrix such that $AA^{-1} = A^{-1}A = I$.

**Defn.** The inverse is not guaranteed to exist. If it exists, then $A$ is *invertible*; otherwise $A$ is *not invertible* or *singular*. 
Fact. If matrix $A$ is invertible, then $Ax = b$ has unique solution $x = A^{-1}b$. 
**Inverse of a \(2 \times 2\) Matrix**

The inverse of a \(2 \times 2\) matrix has formula:

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

The formula also captures when the inverse exists: the matrix is invertible if and only if \(ad - bc \neq 0\).
Obtaining Matrix Inverses by Reduction

One way to find the inverse is to solve the collection of $n$ vector equations $Ax = e_1, \ldots, Ax = e_n$ (where the $e_j$ are the columns of $I_n$ as before). Equivalently:

**ALGOR** To find inverse of matrix $A$, augment with the identity matrix $I_n$, then bring to reduced row echelon form.
Example Inverse Calculation

\[ C = \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix} \text{ is augmented to } \begin{bmatrix} 3 & -5 & 1 & 0 \\ -5 & 9 & 0 & 1 \end{bmatrix} \]

This reduces to \begin{bmatrix} 1 & 0 & 9/2 & 5/2 \\ 0 & 1 & 5/2 & 3/2 \end{bmatrix}

so that \[ C^{-1} = \begin{bmatrix} 9/2 & 5/2 \\ 5/2 & 3/2 \end{bmatrix} \]
**Fact.** If $A$ and $B$ are square matrices of the same size:

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1} A^{-1}$ (Note the reversal!)

(c) $(A^T)^{-1} = (A^{-1})^T$. 

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**Formulas**
Characterization of Invertible Matrices

The big theorem.

**Fact.** For an $n \times n$ matrix $A$, the following are equivalent:

- $A$ is invertible
- $A$ has $n$ pivots
- $A$ is row equivalent to $I_n$
- $Ax = 0$ has a unique solution
- the columns of $A$ are linearly independent
- the columns of $A$ span $\mathbb{R}^n$
- the range of transform $x \mapsto Ax$ is all of $\mathbb{R}^n$
**Block Matrices**

**Defn.** A *partitioned* matrix has the rows and columns partitioned, dividing the matrix up into *blocks*.

For example, if two $n \times n$ matrices are partitioned into $n/2 \times n/2$ blocks and both have a zero block as the bottom left, then

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} D & E \\ 0 & F \end{bmatrix} = \begin{bmatrix} AD & AE + BF \\ 0 & CF \end{bmatrix}$$
**Block-Diagonal Matrices**

**Defn.** A *block-diagonal* matrix is one where all blocks off the diagonal are zero.

**Fact.** A block-diagonal matrix is invertible if and only if all the diagonal blocks are invertible. Moreover, its inverse is the block-diagonal matrix with the inverses of the diagonal blocks.
Example Inverse of Block-Diagonal Matrix

\[ \begin{bmatrix}
6 & 0 & 0 \\
0 & 3 & -5 \\
0 & -5 & 9
\end{bmatrix}^{-1} = \begin{bmatrix}
1/6 & 0 & 0 \\
0 & 9/2 & 5/2 \\
0 & 5/2 & 3/2
\end{bmatrix} \]
**Defn.** A **lower triangular** matrix is one whose entries above the main diagonal are zero. An **upper triangular** matrix is defined similarly.

For example, a diagonal matrix is both lower and upper triangular.

**Fact.** A square triangular matrix is invertible if and only if every entry on the diagonal is nonzero.
The inverse of a square matrix $A$ is the matrix $A^{-1}$ such that their product is the identity. If inverse exists, then $A$ is invertible; otherwise $A$ is singular. If matrix $A$ is invertible, then $Ax = b$ has unique solution $A^{-1}b$.

The inverse of a $2 \times 2$ matrix has formula $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

One way to find the inverse is to augment with the identity matrix and bring to reduced row echelon form.
An $n \times n$ matrix is invertible whenever it has $n$ pivots; equivalently the columns are linearly independent and span $\mathbb{R}^n$.

A block-diagonal matrix has rows and columns partitioned, dividing the matrix up into blocks such that all blocks off the diagonal are zero. Its inverse is determined by the inverses of its diagonal blocks. A lower [upper] triangular matrix is one whose entries above [below] the diagonal are zero.