Properties of Eigenvalues and Eigenvectors
Algebraic Multiplicity

**Defn.** The *algebraic multiplicity* of eigenvalue \( \lambda \) is its multiplicity as a root of the characteristic polynomial.

**Fact.** The dimension of the eigenspace of \( \lambda \) is at most its algebraic multiplicity.
Eigenvectors and Linear Independence

**Fact.** Eigenvectors for distinct eigenvalues are linearly independent.
Eigenvalues of Matrix Powers

**Fact.** If matrix $A$ has eigenvalues $\lambda_i$, then the power $A^k$ has eigenvalues $\lambda_i^k$. Moreover, the eigenvectors are the same.
The **trace** of a matrix is defined as the sum of the diagonal entries.

**Fact.** For any matrix $A$,

(a) the determinant of $A$ equals the product of its eigenvalues.

(b) the trace of $A$ equals the sum of its eigenvalues.
Recall that \( i \) denotes the square-root of \(-1\).

**Defn.** If \( \lambda = a + bi \), then its \textbf{(complex) conjugate} is \( a - bi \).
Complex Eigenvalues

**Fact.** If $\lambda$ is a complex eigenvalue of $A$, then so is its conjugate.
An Example

Consider the matrix \[
\begin{bmatrix}
a & -b \\ b & a
\end{bmatrix}.
\]

The characteristic polynomial is \((a - \lambda)^2 + b^2\); eigenvalues are \(\lambda = a \pm bi\).

As a matrix transform, this represents scaling by \(\sqrt{a^2 + b^2}\) and rotation through \(\arctan \frac{b}{a}\).
Symmetric Matrices

**Fact.** A real symmetric matrix has only real eigenvalues.
The algebraic multiplicity of eigenvalue is its multiplicity as root of the characteristic polynomial; the eigenspace has dimension at most its algebraic multiplicity.

Eigenvectors for distinct eigenvalues are linearly independent.

If $A$ has eigenvalue $\lambda_i$, then $A^k$ has eigenvalue $\lambda_i^k$ with same eigenvector.
The product of the eigenvalues is the determinant; the sum of the eigenvalues is the trace, which is the sum of the diagonal entries.

If $\lambda$ is complex eigenvalue of real matrix, then so is its conjugate. A real symmetric matrix has real eigenvalues.