2 Vectors and Linear Combinations

In this chapter we define vectors and operations involving vectors.

2.1 Vectors

You might have encountered vectors elsewhere. Some books define both a column vector and a row vector, but we will only use the column vector.

> *A (column)* ***vector*** *is a column of numbers.*

Thus we may think of the columns of a matrix as vectors. (Indeed, some books define vectors first and then define a matrix in this way.) We will in general use bold letters for vector variables, such as $\boldsymbol{x}$ and $\boldsymbol{v}$. To save space, we sometimes write the column vector $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ as $(3, 5)$. To be equal, two vectors must have the same size and the same entries in order.

Several vector operations can be defined.

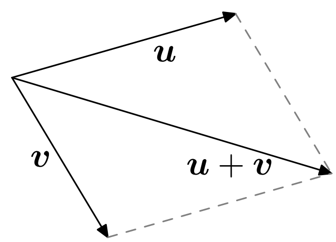
> *Vector* ***addition*** *is performed by adding the corresponding entries. In algebraic notation:*
>
> $$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$
>
> *Similarly,* ***scalar multiplication*** *is performed by scaling each entry. In algebraic notation:*
>
> $$c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c u_1 \\ c u_2 \end{bmatrix}$$

Example.

$$3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix} + \begin{bmatrix} -4 \\ 28 \end{bmatrix} = \begin{bmatrix} 2 \\ 40 \end{bmatrix}$$

We use $\mathbb{R}^d$ for the set of all d-entry vectors whose entries are real numbers. (Complex numbers will make an appearance later.) One can associate vectors in $\mathbb{R}^d$ with the corresponding point in space to give geometric descriptions. For example, $\mathbb{R}^2$ is the 2-dimensional plane. The addition of two vectors in $\mathbb{R}^2$ can be interpreted geometrically:

for example, $(7, 2) + (3, -5) = (10, -3)$ can be viewed as "if you go 7 blocks east and 2 blocks north, then 3 blocks east and 5 blocks south, then you are 10 blocks east and 3 blocks south of where you started. Equivalently, vector addition in $\mathbb{R}^2$ can be interpreted as the diagonal of the parallelogram they create:



These two vector operations obey standard properties of arithmetic. These properties include the commutative law (order of addition does not matter), associative law (brackets do not matter), and distributive laws (for example, $c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$).

2.2 Linear Combinations and Spans

> *A* ***linear combination*** *of vectors is formed by summing some multiple of each vector. The multipliers are called the* ***weights****.*

For example, $3\boldsymbol{u} + 4\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

> *The* ***span*** *of a collection of vectors is the set of all possible linear combinations. If S is a set, we will denote its span by Span(S).*

EXAMPLE. *The span of a single (nonzero) vector is a line. The span of two vectors is usually a plane.*

We define next the product of a matrix with a vector. We use the notation $A = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ to mean that the columns of matrix A are the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

> *If A is an $m \times n$ matrix and vector $\boldsymbol{x}$ is in $\mathbb{R}^n$, then the* ***matrix-vector product*** *$A\boldsymbol{x}$ is defined to be the linear combination of the columns of A specified by $\boldsymbol{x}$. That is, if $A = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$ then*
>
> $$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_n\boldsymbol{x}_n$$

EXAMPLE.

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 47 \end{bmatrix}$$

Every linear system is equivalent to a matrix equation. From the definition of matrix-vector multiplication we immediately obtain the following fact:

Fact 2.1 *The matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is a linear combination of the columns of A.*

In particular: testing whether a vector $\boldsymbol{b}$ is in the span of some collection of vectors, is equivalent to asking whether the augmented matrix with those columns is consistent.

The next fact addresses the question of when a matrix equation for fixed A has a solution for **ALL** possible $\boldsymbol{b}$:

Fact 2.2 *For matrix A, the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$*
$\iff$ *the span of the columns of A is $\mathbb{R}^m$*
$\iff$ *A has a pivot in each row.*

Note that this fact says that three conditions are **equivalent**: either all three conditions hold or none of them hold.

PROOF. *The first and second conditions are equivalent, because the product $A\boldsymbol{x}$ is by definition a linear combination of the columns of A. That is, the first and second conditions are different ways of saying the same thing.*
The main part of the proof is to show that the first and third conditions are equivalent. Assume first that one can pivot in every row. This means that the echelon form cannot have a row that is all-zero outside the augmented column; thus the echelon form will always pass the test for consistent system, and the matrix equation is always consistent.
On the other hand, assume that one cannot pivot in every row. Then the echelon form has zeroes in the bottom row outside the augmented column. Thus one can choose the original constant in that row to ensure that in the augmented column there is a nonzero entry at the end. That is, one can create an inconsistent system by choosing $\boldsymbol{b}$ suitably, which has no solution.

2.3 Homogenous Systems and Parametric Vector Form

*A **homogeneous system** is one like $A\boldsymbol{x} = \mathbf{0}$. It always has at least the **trivial solution** $\boldsymbol{x} = \mathbf{0}$.*

Solving the homogeneous system is part of our approach to solving the general system. Indeed, one feature of vectors is that they provide another way to express the solution to a linear system.

ALGOR *The solution set to a general linear system can be written in **parametric vector form** as: one vector plus an arbitrary linear combination of vectors satisfying the corresponding homogeneous system.*

For example, the solution set to the homogeneous system could be a plane through the origin, while the solution set to a general linear system could be a plane shifted.

Example. *Consider the following reduced row echelon form*

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Before, we saw that x_2 and x_4 were free variables, and wrote the solution as

$$\begin{aligned} x_1 &= -1 - 2x_2 + 3x_4 \\ x_3 &= 5 - x_4 \end{aligned}$$

This can be reformulated in vector notation by adding the ("silly") equations that say the free variables equal themselves. Here this means

$$\begin{aligned} x_1 &= -1 - 2x_2 + 3x_4 \\ x_2 &= x_2 \\ x_3 &= 5 - x_4 \\ x_4 &= x_4 \end{aligned}$$

And thus this system has solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

2.4 Linear Independence of Vectors

We consider next a crucial, but slippery, concept.

A collection of vectors is ***linearly independent*** *if the only linear combination of them that equals* $\mathbf{0}$ *is the trivial combination (all weights zero). Otherwise it is said to be* ***linearly dependent****.*

Example. *The trio* $\{(0,1,1),(1,2,-3),(-2,4,14)\}$ *is linearly dependent since* $6(0,1,1) - 2(1,2,-3) - 1(-2,4,14) = (0,0,0)$. *On the other hand, the trio* $\{(1,0,0),(0,1,0),(0,0,1)\}$ *is linearly independent, as the only linear combination that produces the zero-vector is* 0 *of each vector.*

We express two important examples as a fact:

Fact 2.3 *(a) A pair of vectors is linearly dependent if and only if one vector is a multiple of the other.*
(b) If a set contains the zero vector, then it is linearly dependent.

More generally, a collection is linearly dependent if at least one vector in the collection can be written as a linear combination of the other vectors.

Example. *If vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are linearly independent, then they span a plane through the origin. Further, inserting* $\boldsymbol{w}$ *into the collection produces a linearly independent set if and only if* $\boldsymbol{w}$ *is not in* $Span(\{\boldsymbol{u},\boldsymbol{v}\})$.

We will use the concept of linear independence often in the course. The first application is to note that this concept captures when a homogenous system has a unique solution.

Fact 2.4 *The columns of matrix* A *are linearly independent*
$\iff$ $A\boldsymbol{x} = \mathbf{0}$ *has only the trivial solution*
$\iff$ *there is no free variable*

Proof. *The first and second condition are equivalent by the definitions of matrix-vector multiplication and linear independence. Our algorithm for solving linear systems shows that the second and third condition are equivalent.*

One consequence of the above fact, is that if there are more columns than rows, then the columns must be linearly dependent.

Practice

2.1. Consider the matrix

$$B = \begin{bmatrix} 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) How many rows of B contain a pivot?

(b) Is the span of the columns of B all of $\mathbb{R}^4$?

(c) Give one nonzero vector $\boldsymbol{d}$ such that $B\boldsymbol{x} = \boldsymbol{d}$ has a solution.

(d) Give one nonzero vector $\boldsymbol{d}$ such that $B\boldsymbol{x} = \boldsymbol{d}$ does not have a solution.

2.2. Give the solution set in parametric vector form for the following systems:

$$\begin{aligned} a + 2b - c + d &= 2 \\ 2a \qquad - c + 2d &= 8 \\ 4a + 4b + 5c - 4d &= 12 \end{aligned} \qquad \begin{aligned} x + 2y + 3z + 4t &= 3 \\ 2x + 4y + 7z \qquad &= -5 \\ 3x + 6y + 10z + 4t &= -2 \end{aligned}$$

2.3. For each triple of vectors, find all value(s) h for which the triple of vectors is linearly dependent:

(a) $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, h)$

(b) $(1, -1, 1)$, $(2, 3, 4)$, $(5, h, h)$

(c) $(0, 0, 0)$, $(1, h, 3)$, $(4, -5, -2)$

Solutions to Practice Exercieses

2.1. (a) 3 (2nd, 4th, 5th)

(b) No.

(c) Any vector where last entry is zero.

(d) Any vector where last entry is nonzero.

2.2. The reductions proceed:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 0 & -1 & 2 & 8 \\ 4 & 4 & 5 & -4 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & -4 & 1 & 0 & 4 \\ 0 & -4 & 9 & -8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & -4 & 1 & 0 & 4 \\ 0 & 0 & 8 & -8 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & -4 & 0 & 1 & 4 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1/2 & 4 \\ 0 & 1 & 0 & -1/4 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 2 & 4 & 7 & 0 & -5 \\ 3 & 6 & 10 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 1 & -8 & -11 \\ 0 & 0 & 1 & -8 & -11 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 1 & -8 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 28 & 36 \\ 0 & 0 & 1 & -8 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the solution sets are:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ -11 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -28 \\ 0 \\ 8 \\ 1 \end{bmatrix}$$

2.3. (a) $h = 0$

(b) If third vector is a times first plus b times second, need $a + 2b = 5$ and $-a + 3b = a + 4b$. Solves to $a = -5/3$ and $b = 10/3$, and so $h = 35/3$.

(c) All h.