4 Determinants

Associated with each square matrix is a number called its determinant. The most important property is that the determinant is zero precisely when the matrix is not invertible. We give below a general definition of a determinant. For calculations, however, the formulas and results in the later sections are more useful. (And indeed, some books use those formulas as the definition.)

4.1 Introduction to Determinants

Determinants are defined for square matrices. The determinant of square matrix A is denoted $\det A$, or indicated by the use of vertical lines replacing the square brackets of the matrix.

Before we give the full definition, let us note that the **determinant** of a 2×2 matrix has a famous formula:

Fact 4.1

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We saw the expression $ad - bc$ earlier in the formula for the inverse of a 2×2 matrix. Indeed, like in the 2×2 case, we will see that the determinant being nonzero captures when a matrix is invertible.

4.2 A Definition of Determinant

The determinant of a general matrix can be defined in terms of permutation matrices.

*A **permutation matrix** is a square matrix that contains only 0's and 1's with exactly one 1 in each row and column*

EXAMPLE. *The identity matrix is a permutation matrix. So too is*

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

There are exactly $n!$ permutation matrices of size $n \times n$.

The ***sign*** *of a permutation matrix is $(-1)^k$, where k is the number of row interchanges needed to change the matrix to be the identity.*

EXAMPLE. *The identity matrix has sign $+1$. The matrix P of the previous example has sign of $+1$, since it can be made the identity by first interchanging the first and third rows, then interchanging the third and fourth row.*

With this machinery, we are now able to define the determinant.

The ***determinant*** *of an $n \times n$ matrix A is defined by:*

- *consider each $n \times n$ permutation matrix P*
- *for each P, multiply together the corresponding entries of A (to obtain what we call a* ***transversal****)*
- *finally sum each transversal after multiplying it by the sign of its P*

For example, in the 2×2 case, there are two permutation matrices $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then P corresponds to the transversal ad and has positive sign, while Q corresponds to the transversal bc and has negative sign. Thus we get the formula for the determinant of a 2×2 matrix above (Fact 4.1.).

Though we will usually not evaluate determinants using this formula, it does reveal some properties. For example, if every entry in some row is zero, then each of the transversals contains a zero, and so is zero. It follows that:

Fact 4.2 *If matrix A has an all-zero row or column, then* $\det A = 0$.

Here is another useful property:

Fact 4.3 *The determinant of a triangular matrix is the product of the diagonal entries.*

PROOF. *Every transversal except the main diagonal is guaranteed to contain a 0-entry and thus be 0. The diagonal comes from the identity permutation matrix, which has positive sign (since $k = 0$).*

In particular, the determinant of the identity matrix I is $\det I = 1$.

4.3 Recursive Formula: Cofactor Expansion

It can be shown that the definition of determinant implies the following formula.

Fact 4.4 *Assume A is an $n \times n$ matrix. Let A_{ij} denote the matrix formed by removing row i and column j. Then* ***expansion*** *across the first row of A gives the formula*

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}$$

The formula is called **recursive** because it give the value in terms of the value for smaller versions of the same problem. The idea behind the proof of the formula (which we omit) is that if one considers a transversal of A that contains a_{11}, then the rest of it is a transversal of A_{11}; and thus the contribution to the overall determinant involving a_{11} is given by $a_{11} \det A_{11}$.

EXAMPLE. *Let B be the following matrix:*

$$\begin{bmatrix} 3 & 6 & 0 \\ 2 & 7 & -1 \\ 0 & 4 & -8 \end{bmatrix}$$

Then B has determinant

$$3 \begin{vmatrix} 7 & -1 \\ 4 & -8 \end{vmatrix} - 6 \begin{vmatrix} 2 & -1 \\ 0 & -8 \end{vmatrix} + 0 \begin{vmatrix} 2 & 7 \\ 0 & 4 \end{vmatrix} = 3 \times (-52) - 6 \times (-16) + 0 = -60.$$

One can expand across other rows, but note that the sign is always $(-1)^{i+j}$. Each term $C_{ij} = (-1)^{i+j} A_{ij}$ in the expansion is called a **cofactor**.

4.4 Properties of Determinants

As we mentioned at the start, the most important aspect of determinants is the fundamental fact:

Fact 4.5 *A square matrix is invertible if and only if its determinant is not zero.*

This is equivalent to saying that the determinant is zero if and only if the columns (and rows) are linearly dependent.

This result is a consequence of the more general fact that the elementary row operations do not change the determinant much. We omit the proof of the following:

Fact 4.6
- ≻ *Adding a multiple of a row to another row does not change* det
- ≻ *Interchanging two rows flips the sign of* det
- ≻ *Multiplying a row by a scalar does the same to* det

By the result about triangular matrices earlier (Fact 4.3), the above result gives us another method to calculate the determinant:

ALGOR *Assume A is an $n \times n$ matrix. If one obtains pivots in every row/column when reducing A to echelon form, without using an interchange, then the determinant of A is the product of the pivots.*

Example. *Consider the matrix B from earlier.*

$$\begin{bmatrix} 3 & 6 & 0 \\ 2 & 7 & -1 \\ 0 & 4 & -8 \end{bmatrix}$$

This reduces to

$$\begin{bmatrix} 3 & 6 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & -\frac{20}{3} \end{bmatrix}$$

without interchanges. Thus the determinant of B is $3 \times 3 \times (-\frac{20}{3}) = -60$.

Other properties of determinants include the following. We omit the proofs:

Fact 4.7 *(a)* $\det(A^T) = \det A$
(b) $\det(AB) = (\det A)(\det B)$
(c) $\det(A^{-1}) = \frac{1}{\det A}$

4.5 Applications of Determinants

Though we don't give it or use it, a famous idea is called **Cramer's rule**. This says that the solution $\boldsymbol{x}$ to the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ can be expressed in terms of the determinants of A and matrices built from A. There is similarly a formula for the inverse A^{-1}.

There is also a geometric interpretation of the determinant. The **volume** of a box whose sides are vectors is given by the absolute value of the associated determinant. For example, the area of the parallelogram determined by vectors (x_1, y_1) and (x_2, y_2) is $|x_1y_2 - x_2y_1|$.

Fact 4.8 *In $\mathbb{R}^2$, if one applies a matrix transform M to some shape, then the area of the shape changes by a factor of* $\det M$.

A similar result holds for the change in volumes under matrix transforms in $\mathbb{R}^3$.

Practice

4.1. Calculate the determinants of the following matrices using cofactors.

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 4 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad L = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 1 & 5 \end{bmatrix}$$

4.2. Calculate the determinants of the matrices in the previous question using row reduction.

4.3. Suppose A is a 3×3 matrix such that $\det A = 5$. Give the determinant of:

(a) A^T

(b) $(A^2)^{-1}$

(c) The matrix that results if one takes A and **replaces** the 2nd row by the sum of the 1st and 3rd rows.

(d) The matrix that results if one takes A and **increases** the 2nd row by the sum of the 1st and 3rd rows.

(e) $A + A$.

(f) $A + A^T$.

Solutions to Practice Exercieses

4.1. $\det C = 30$
$\det D = 9$
$\det L = -10$

4.2. $\det C = 30$
$\det D = 9$
$\det L = -10$

4.3. (a) 5 (by Fact 4.7)
(b) $\frac{1}{25}$ (by Fact 4.7)
(c) 0 (the rows of resultant matrix are linearly dependent so it is not invertible)
(d) 5 (by Fact 4.6)
(e) 40 (equivalent to multiplying each row by 2)
(f) Could be anything.