5 Subspaces, Bases, and Dimension

In this chapter we introduce a subspace, which is a collection of vectors such that if one adds two vectors or scales a vector, one gets a vector already in the collection. We then consider a basis, with which one can construct every vector in a subspace in a unique way; and the size of the basis is the dimension of the subspace.

5.1 Sets Closed Under Operations

A key mathematical idea that we need is the idea of a set closed under an operation.

*A set S is **closed** under some operation if applying that operation to elements of S always produces an element of S.*

Note that the definition says nothing about what happens if you start with two elements outside S; it only says that if you start inside S you cannot escape S using that operation.

EXAMPLE. *The integers are closed under addition: adding two integers always produces an integer. The integers are also closed under subtraction and multiplication: subtracting or multiplying two integers always produces an integer. But the integers are not closed under division.*

EXAMPLE. *The set of positive real numbers is not closed under subtraction: for example, $2 - \pi$ is not positive. This set is closed under addition, multiplication, division, and exponentiation.*

Fact 5.1 *The set of solutions to the homogeneous equation $A\boldsymbol{x} = \mathbf{0}$ is closed under both addition and scalar multiplication.*

In other words, if one takes two solutions and adds them, then the result is again a solution; and if one takes a solution and scales it, then the result is again a solution.

PROOF. *Assume $\boldsymbol{x}$ and $\boldsymbol{x}'$ are solutions to the homogeneous equation. Then $A(\boldsymbol{x} + \boldsymbol{x}') = A\boldsymbol{x} + A\boldsymbol{x}' = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so that $\boldsymbol{x} + \boldsymbol{x}'$ is a solution. Similarly, $A(c\boldsymbol{x}) = c(A\boldsymbol{x}) = c\mathbf{0} = \mathbf{0}$, so that $c\boldsymbol{x}$ is a solution.*

5.2 Subspaces

A ***subspace*** *of* $\mathbb{R}^n$ *is a subset* S *such that*

(0) S *contains the zero vector;*

(1) S *is closed under addition;*

(2) S *is closed under scalar multiplication.*

Consider $\mathbb{R}^2$.

Example. *Let* S *be the set of points* (x, y) *in* $\mathbb{R}^2$ *on the line* $y = 3x$. *This is a subspace. Check the three conditions. But the set of points on the line* $y = 3x + 1$ *is not a subspace (as it satisfies none of the three conditions!)*

Example. *Let* T *be the set of points* (x, y) *in* $\mathbb{R}^2$ *such that* $|x| = |y|$. *This is not a subspace. Though the set* T *satisfies Conditions (0) and (2) (check!), it is not closed under addition: for example* $(2, -2)$ *and* $(3, 3)$ *are in* T *but their sum* $(5, 1)$ *is not in* T.

Example. *Let* U *be the set of points* (x, y) *in* $\mathbb{R}^2$ *such that* $x, y \geq 0$. *This is not a subspace. Though the set* U *satisfies Conditions (0) and (1) (check!), it is not closed under scalar multiplication: for example* $(2, 3)$ *is in* U, *but scaling by* -1 *produces* $(-2, -3)$, *which is not in* U.

By the matrix-inversion theorem, we know that in $\mathbb{R}^2$ that if two vectors are linearly independent, then the span of the two vectors in all of $\mathbb{R}^2$. Thus we get

Fact 5.2 *Every subspace of* $\mathbb{R}^2$ *is either* $\{0\}$, *a line through the origin, or* $\mathbb{R}^2$ *itself.*

Example. *Consider* $\mathbb{R}^3$ *and let* S *be the set of all vectors whose third coordinate is* 0. *This is a subspace. Because, the zero vector is in* S*; adding two vectors in* S *produces a vector in* S*; and scaling a vector in* S *produces a vector in* S. *On the other hand, let* T *be the set of all vectors whose third coordinate is* 1*; this is not a subspace as it fails all three conditions.*

There is a result for $\mathbb{R}^3$ that is similar to the result for $\mathbb{R}^2$:

Fact 5.3 *Every subspace of* $\mathbb{R}^3$ *is either* $\{0\}$, *a line through the origin, a plane through the origin, or* $\mathbb{R}^3$ *itself.*

Here is an important example:

EXAMPLE. *The set containing just the zero-vector is always a subspace.* $\mathbb{R}^n$ *itself is also a subspace.*

One useful fact is the following:

Fact 5.4 *If S is a finite set of vectors in $\mathbb{R}^n$, then Span(S) is a subspace.*

PROOF. *Say $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$. Then $\mathbf{0}$ is the linear combination $0\boldsymbol{v}_1 + \ldots + 0\boldsymbol{v}_k$. So Condition (0) holds. If $\boldsymbol{x} = a_1\boldsymbol{v}_1 + \ldots + a_k\boldsymbol{v}_k$ and $\boldsymbol{y} = b_1\boldsymbol{v}_1 + \ldots + b_k\boldsymbol{v}_k$, then $\boldsymbol{x} + \boldsymbol{y} = (a_1 + b_1)\boldsymbol{v}_1 + \ldots + (a_k + b_k)\boldsymbol{v}_k$. Thus Span$(S)$ is closed under addition. Also, $c\boldsymbol{x} = (ca_1)\boldsymbol{v}_1 + \ldots + (ca_k)\boldsymbol{v}_k$. So Span$(S)$ is also closed under scalar multiplication.*

(The above fact is also true about infinite S; indeed the proof is the same except one has to use different notation.)

5.3 Bases

*A **basis** B of a subspace S is a set such that*

(i) B is linearly independent, and

(ii) the span of B is all of S.

The set $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ forms what is called the **standard basis** of $\mathbb{R}^n$. (Recall that $\boldsymbol{e}_i$ denotes the i^{th} column of the identity matrix.)

EXAMPLE. *Consider $\mathbb{R}^2$. The standard basis is $\{(1, 0), (0, 1)\}$. But $\mathbb{R}^2$ has many bases. In fact, if one takes any two vectors that are linearly independent, then their span is $\mathbb{R}^2$ and thus the pair of vectors is a basis. On the other hand, any trio of vectors is linearly dependent, while no single vector has a span that is $\mathbb{R}^2$.*

By the Matrix-Inversion Theorem (Fact 3.6) we get the following important fact:

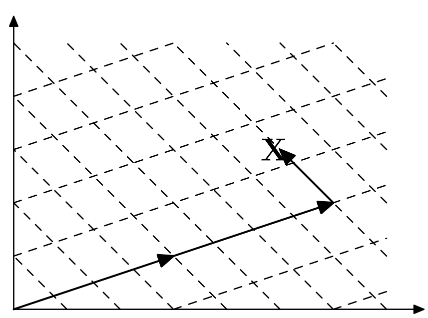
EXAMPLE. *If A is an invertible $n \times n$ matrix, then its columns form a basis of $\mathbb{R}^n$.*

The usefulness of a basis is the following fundamental result:

Fact 5.5 *If B is a finite basis for subspace S, then every element in S is* ***uniquely*** *expressible as a linear combination of vectors of B.*

Proof. *Say $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k\}$. Let $\boldsymbol{w}$ be any vector in S. Because the span of B is S, we know that $\boldsymbol{w}$ can be expressed as a linear combination of vectors of B. Consider any two linear combinations that give $\boldsymbol{w}$; say $\boldsymbol{w} = c_1\boldsymbol{b}_1 + \ldots + c_k\boldsymbol{b}_k$ and $\boldsymbol{w} = d_1\boldsymbol{b}_1 + \ldots + d_k\boldsymbol{b}_k$. Then, by subtracting the one equation from the other, we get that $\mathbf{0} = (c_1 - d_1)\boldsymbol{b}_1 + \ldots + (c_k - d_k)\boldsymbol{b}_k$. Because B is linearly independent, it must be that $(c_1 - d_1) = \cdots = (c_k - d_k) = 0$. That is, $c_1 = d_1$, $c_2 = d_2$, etc. It follows that the two linear combinations for $\boldsymbol{w}$ are the same. Or in other words, the linear combination is unique.*

In $\mathbb{R}^2$, one can think of the linear combination as identifying a point in the plane: with the standard basis, $(5, 3)$ means go 5 units in $\boldsymbol{e}_1$ direction and (then) 3 units in $\boldsymbol{e}_2$ direction. But one can also uniquely identify the point given any two linearly independent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$: go so many units in the direction of $\boldsymbol{u}$ and then so many units in the direction of $\boldsymbol{v}$. For example, $(5, 3)$ can also be reached by going 2 units of $(3, 1)$ and -1 units of $(1, -1)$. Instead of a rectangular grid, one can think of grid lines parallel to the vectors:



5.4 Dimension

The big theorem is the following result. We omit the proof.

Fact 5.6 *All bases of a subspace have the same number of elements.*

Thus we can define:

*The **dimension** of a subspace is defined to be the number of elements in a basis.*

EXAMPLE. $\mathbb{R}^d$ *has dimension* d.
A line (through the origin) has dimension 1.
A place (through the origin) has dimension 2.

An important but silly example is:

EXAMPLE. *The subspace* $\{\mathbf{0}\}$ *has dimension* 0 *and an empty basis. (We noted earlier that any set containing the zero-vector is linearly dependent. The claim that* $\mathbf{0}$ *is in the span of the empty set is mathematical prestidigitation: one could take it simply as a definition, but the argument is that if you add up nothing you get the zero-vector. . .)*

It can be shown that:

Fact 5.7 *Assume subspace* S *has dimension* p.
(a) Any linearly independent set of p *vectors forms a basis of* S.
(b) Any set of p *vectors whose span is* S *forms a basis of* S.

In other words, a basis is simultaneously a spanning set that is as small as possible, and a linearly independent set that is as large as possible.

Practice

5.1. Consider the following subsets of $\mathbb{R}^3$. Explain why each is not a subspace.

(a) The points in the xy-plane in the first quadrant.

(b) All integer solutions to the equation $x^2 + y^2 = z^2$.

(c) All points on the line $x + z = 5$.

(d) All vectors where the three coordinates are the same in absolute value.

5.2. What is the dimension of:

(a) The set of all vectors in $\mathbb{R}^{26}$ whose last coordinate is 0.

(b) The set of all vectors in $\mathbb{R}^{26}$ that are a multiple of $(1, 1, \ldots, 1)$.

(c) The set of all solutions to the equation $a + b + c + \ldots + z = 0$.

5.3. Show that the intersection of two subspaces is always a subspace.

Solutions to Practice Exercieses

5.1. (a) Not closed under scalar multiplication: $(1,1,0)$ in set but not $-1(1,1,0)$

(b) Not closed under scalar multiplication: $(3,4,5)$ in set but not $\frac{1}{2}(3,4,5)$

(c) Does not contain zero.

(d) Not closed under addition: $(1,1,1)$ and $(1,1,-1)$ in set but not their sum $(2,2,0)$.

5.2. (a) 25

(b) 1

(c) 25

5.3. Consider subspaces S and T. Since the zero vector is in both, it's in their intersection. Consider vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in the intersection. Since both S and T are closed under addition, the sum $\boldsymbol{u}+\boldsymbol{v}$ is in both and therefore in their intersection. Similarly one gets closure under scalar multiplication.