6 Bases and Dimension

A basis is a set of vectors that one can use to build all of a vector space, but where every vector in the basis is needed. It turns out that every basis has the same size: this we call the dimension of the vector space.

6.1 Bases

A basis $B$ of a vector space is a set such that

(i) $B$ is linearly independent, and

(ii) the span of $B$ is the whole space.

The set $\{e_1, \ldots, e_n\}$ forms what is called the standard basis of $\mathbb{R}^n$.

Example. Consider $\mathbb{R}^2$. The standard basis is $\{(1,0), (0,1)\}$. But $\mathbb{R}^2$ has many bases. In fact, if one takes any two vectors that are linearly independent (that is, neither is a multiple of the other), then their span is the space and thus the pair of vectors is a basis. On the other hand, any trio of vectors is linearly dependent, while no one vector has a span that is the whole space.

By the Matrix-Inversion Theorem (Fact 3.6) we get the following important fact:

Example. If $A$ is an invertible $n \times n$ matrix, then its columns form a basis of $\mathbb{R}^n$.

Example. The set $\{1, t, t^2, \ldots, t^n\}$ is a basis for the space $\mathbb{P}_n$ of polynomials.

It can be shown that:

Fact 6.1  (a) Every spanning set contains a basis.
(b) Every linearly independent set can be extended to a basis.

Indeed, a basis is simultaneously a spanning set that is as small as possible, and a linearly independent set that is as large as possible. The usefulness of a basis is the following fundamental result:

Fact 6.2  If $B$ is a finite basis for vector space $V$, then every element in $V$ is uniquely expressible as a linear combination of vectors of $B$. 
Proof. Say \( B = \{b_1, \ldots, b_k\} \). Let \( w \) be any vector in \( V \). Because the span of \( B \) is \( V \), we know that \( w \) can be expressed as a linear combination of vectors of \( B \).

Consider any two linear combinations that give \( w \); say \( w = c_1 b_1 + \ldots + c_k b_k \) and \( w = d_1 b_1 + \ldots + d_k b_k \). Then, by subtracting the one equation from the other, we get that \( 0 = (c_1 - d_1)b_1 + \ldots + (c_k - d_k)b_k \). Because \( B \) is linearly independent, it must be that \( (c_1 - d_1) = \cdots = (c_k - d_k) = 0 \). That is, \( c_1 = d_1, c_2 = d_2, \) etc. It follows that the two linear combinations for \( w \) are the same. Or in other words, the linear combination is unique.

(The above result also works for a basis is infinite; the proof just needs some more notation.)

In \( \mathbb{R}^2 \), one can think of the linear combination as identifying a point in the plane: with the standard basis, \((5,3)\) means go 5 units in \( e_1 \) direction and (then) 3 units in \( e_2 \) direction. But we can also uniquely identify the point given any two linearly independent vectors \( u \) and \( v \): go so many units in the direction of \( u \) and then so many units in the direction of \( v \). For example, \((5,3)\) can also be reached by going 2 units of \((3,1)\) and \(-1\) units of \((1,-1)\). Instead of a rectangular grid, one can think of grid lines parallel to the vectors:

\[\text{graph image}\]

### 6.2 Dimension

The big theorem is the following result. We omit the proof.

**Fact 6.3** All bases of a given vector space have the same number of elements.

Thus we can define:

**The dimension** of a vector space is defined to be the number of elements in a basis.
Example. $\mathbb{R}^d$ has dimension $d$.

Example. $\mathbb{P}_n$ has dimension $n + 1$. As noted above, a basis is the set $\{1, t, t^2, \ldots, t^n\}$.

Example. The set $C[t]$ of continuous functions has infinite dimension.

Example. The set $M_2$ of $2 \times 2$ matrices has dimension 4. A basis is, for example, the quartet of $2 \times 2$ matrices that have three 0’s and one 1. (Equivalently, to specify a $2 \times 2$ matrix one has to give 4 numbers.)

Example. In calculus one might consider the equation $f'' = -f$; that is, all functions whose second derivative is the negative of the function. One solution to this equation is $\sin t$; another is $\cos t$. Indeed, any linear combination of $\sin t$ and $\cos t$ is a solution. It can be shown that this gives all the solutions. That is, the set of all functions $f(t)$ that obey the differential equation has basis $\{\sin t, \cos t\}$.

An important but silly example is:

Example. The space $\{0\}$ has dimension 0 and an empty basis. (We noted earlier that any set containing the zero-vector is linearly dependent. The claim that 0 is in the span of the empty set is mathematical prestidigitation: one could take it simply as a definition, but the argument is that if you add up nothing you get the zero-vector…)

Facts 6.1 and 6.3 imply that:

**Fact 6.4** Assume vector space $V$ has dimension $p$.
(a) Any linearly independent set of $p$ vectors forms a basis of $V$.
(b) Any set of $p$ vectors whose span is $V$ forms a basis of $V$.

### 6.3 Bases for the Three Matrix Subspaces

We have already seen how to construct a basis for the null space. We did that when finding the general solution to the homogeneous system $Ax = 0$ in parametric vector form. That is,

**ALGOR** Basis for Null Space.
A basis for $\text{Nul } A$ is obtained by creating one vector for each free variable after row reduction. The dimension of the null space is the number of free variables.
6 BASES AND DIMENSION

The situation for the column space is the following. We need a set of linearly independent columns that cannot be increased. This set can be determined by row reduction, even though row reduction changes the column space:

**ALGOR Basis for Column Space.**

A basis for Col A is obtained by taking each original column vector that corresponds to a pivot column. The dimension of the column space is the number of basic variables.

Since row operations do not change the row space, we get that

**ALGOR Basis for Row Space.**

A basis for the row space is obtained by taking the nonzero rows of echelon form. The dimension of the row space is the number of pivots.

**Example.**

Matrix

\[
\begin{bmatrix}
1 & -1 & 4 & 7 & -1 \\
5 & -5 & 0 & 15 & 11 \\
2 & -2 & -2 & 4 & 6
\end{bmatrix}
\]

reduces to

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & 11/5 \\
0 & 0 & 1 & 1 & -4/5 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So the null space has dimension 3. Further, by our discussion of parametric vector description, the null space has a basis of \((1,1,0,0,0)\), \((-3,0,-1,1,0)\), and \((-11/5,0,4/5,0,1)\).

The column space has dimension 2 with a basis of \((1,5,2)\) and \((4,0,-2)\).

The row space has dimension 2 with a basis of \((1,-1,0,3,11/5)\) and \((0,0,1,1,-4/5)\).

**Example.** For matrix

\[
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

the null space is \(\{0\}\), the column space is all of \(\mathbb{R}^2\), as is the row space.

In particular, the above discussion shows:

**Fact 6.5** For any matrix, its row and column spaces have the same dimension.

The rank of a matrix A is the dimension of Col A or equivalently the dimension of Row A.

We can add another characterization of invertible matrices:
Fact 6.6  Square matrix $A$ is invertible $\iff$ Nul $A = \{0\} \iff A$ has full rank.

Practice

6.1. In Exercise 5.2 we asked which of the following are vector spaces. For the cases where the set is a space, give the dimension and a basis.

(a) the set of all polynomials with degree exactly 1
(b) the set of all $2 \times 2$ matrices with determinant 2
(c) the set of all diagonal $3 \times 3$ matrices
(d) the set of all vectors in $\mathbb{R}^4$ whose entries sum to 0
(e) the set of all antiderivatives of $f(x) = x^5$

6.2. Consider the matrix

\[
F = \begin{bmatrix}
2 & -1 & 0 & -3 \\
12 & -6 & 1 & 7 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) Give the dimension and a basis of the column space of $F$.
(b) Give the dimension and a basis of the null space of $F$.
(c) Give the dimension and a basis of the row space of $F$.

6.3. Assume $A$ and $B$ are $2 \times 2$ matrices. For each of the following, either prove or disprove. (That is, either explain why we know it’s always true, or give an example that shows it is not always true.)

(a) If both $A$ and $B$ have rank 2, then so does the product $AB$.
(b) If both $A$ and $B$ have rank 1, then so does the product $AB$.
(c) If both $A$ and $B$ have rank 0, then so does the product $AB$.

Solutions to Practice Exercises

6.1. (c) Dimension 3. The trio of matrices with 1 in one diagonal position and 0’s elsewhere.
(d) Dimension 3. For example, $(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)$.
6.2. (a) Dimension 2. Example basis: the first and third columns of $F$.

(b) Dimension 2. Example basis: \[
\begin{pmatrix}
\frac{1}{2} \\
1 \\
0 \\
0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\frac{3}{2} \\
0 \\
-25 \\
1
\end{pmatrix}
\]

(c) Dimension 2. Example basis: the first two rows of $F$.

6.3. (a) True. For $2 \times 2$ matrices, having rank 2 is equivalent to being invertible, and we know from Chapter 3 that the product of invertible matrices is invertible.

(b) Not necessarily true. Consider \[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\] They have rank 1 but their product has rank 0.

(c) True. The only matrix that has rank 0 is the all-zero matrix.