6 The Three Matrix Spaces and Coordinate Systems

In this chapter we consider three standard subspaces associated with a matrix. We also examine how a basis for a subspace can be used to provide a coordinate system.

6.1 The Three Matrix Spaces

For a matrix A, we define three fundamental sets as follows.

- *The **null space** of matrix A, denoted Nul A, is the set of all solutions to the homogeneous system $A\boldsymbol{x} = \mathbf{0}$. That is, all vectors mapped to $\mathbf{0}$ by the matrix transform $x \mapsto Ax$.*
- *The **column space** of matrix A, denoted Col A, is the set of all linear combinations of columns of A.*
- *The **row space** of matrix A, denoted Row A, is the set of linear combinations of rows of A.*

EXAMPLE. *Consider the matrix* $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. *The column space consists of* $(1, 2)$, $(3, 4)$, *and* $(5, 6)$ *and every linear combination thereof (which turns out to be all of* $\mathbb{R}^2$*). The row space consists of* $(1, 2, 3)$ *and* $(4, 5, 6)$ *and every linear combination thereof. The null space contains vectors such as* $(1, -2, 1)$.

All three of these sets are subspaces:

Fact 6.1 *If A is an $m \times n$ matrix, then*
(a) Nul A is a subspace of $\mathbb{R}^n$.
(b) Col A is a subspace of $\mathbb{R}^m$.
(c) Row A is a subspace of $\mathbb{R}^n$.

PROOF. *Parts (b) and (c) follow from the earlier observation that any span is a subspace (Fact 5.4). To prove (a), by the definition we need to check the three conditions. The two closure conditions were noted in Fact 5.1. Further, the zero vector* $\mathbf{0}$ *is in the null space, since* $A\mathbf{0} = \mathbf{0}$.

Our earlier discussion about row operations showed that:

Fact 6.2 *If two matrices are row equivalent, then they have the same row space.*

6.2 Bases for the Three Matrix Spaces

We have actually already seen how to construct a basis for the null space. We did that when finding the general solution to the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ in parametric vector form. That is,

ALGOR BASIS FOR NULL SPACE.
A basis for Nul A is obtained by creating one vector for each free variable after row reduction. The dimension of the null space is the number of free variables.

The situation for the column space is the following. We need a set of linearly independent columns that cannot be increased. This set can be determined by row reduction, even though row reduction changes the column space:

ALGOR BASIS FOR COLUMN SPACE.
A basis for Col A is obtained by taking each ***original*** *column vector that corresponds to a pivot column. The dimension of the column space is the number of basic variables.*

Since row operations do not change the row space, we get that

ALGOR BASIS FOR ROW SPACE.
A basis for the row space is obtained by taking the nonzero rows of echelon form. The dimension of the row space is the number of pivots.

EXAMPLE.

$$\text{Matrx} \quad \begin{bmatrix} 1 & -1 & 4 & 7 & -1 \\ 5 & -5 & 0 & 15 & 11 \\ 2 & -2 & -2 & 4 & 6 \end{bmatrix} \quad \text{reduces to} \quad \begin{bmatrix} 1 & -1 & 0 & 3 & 11/5 \\ 0 & 0 & 1 & 1 & -4/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the null space has dimension 3*. Further, by our discussion of parametric vector description, the null space has a basis of* $(1, 1, 0, 0, 0)$, $(-3, 0, -1, 1, 0)$, *and* $(-11/5, 0, 4/5, 0, 1)$.

The column space has dimension 2 *with a basis of* $(1, 5, 2)$ *and* $(4, 0, -2)$.

The row space has dimension 2 with a basis of $(1, -1, 0, 3, 11/5)$ *and* $(0, 0, 1, 1, -4/5)$.

EXAMPLE. *For matrix* $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, *the null space is* $\{\mathbf{0}\}$, *the column space is all of* $\mathbb{R}^2$, *as is the row space.*

In particular, the above discussion shows:

Fact 6.3 *For any matrix, its row and column spaces have the same dimension.*

The ***rank*** *of a matrix* A *is the dimension of* $Col\, A$ *or equivalently the dimension of* $Row\, A$. *A square matrix has* ***full rank*** *if its rank is equal to the number of rows/columns.*

We can add another characterization of invertible matrices.

Fact 6.4 *Square matrix* A *is invertible* $\iff$ $Nul\, A = \{0\}$ $\iff$ A *has full rank.*

6.3 Coordinate Systems

In Fact 5.5 we noted that if B is a basis for a subspace, then every vertex in the subspace is uniquely expressed as a linear combination of B. These coefficients provide a way to specify the vector:

If B *is a basis of* $\mathbb{R}^n$, *the notation* $[\boldsymbol{x}]_B$ *is defined to be the coefficients used when expressing* $\boldsymbol{x}$ *as a linear combination of vectors in* B. *This vector is called the* ***coordinates*** *of* $\boldsymbol{x}$ *relative to* B.

EXAMPLE. *Consider* $\mathbb{R}^2$ *and basis* B *is* $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$. *If* $\boldsymbol{x} = (5, -6)$, *then* $[\boldsymbol{x}]_B = (7, -2)$. *This can be checked by calculating* $7 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and seeing that this equals* $\begin{bmatrix} 5 \\ -6 \end{bmatrix}$.

6.4 Changing the Basis

One common task is to convert between coordinate systems.

*In $\mathbb{R}^n$, the **change-of-coordinates matrix** P_B has B as its columns.*

Fact 6.5 *If B is a basis of $\mathbb{R}^n$ then*

$$\boldsymbol{x} = P_B\,[\boldsymbol{x}]_B$$

That is, one can convert from coordinates relative to B to standard coordinates by multiplying by the matrix P_B. On the other hand, to convert from standard coordinates to those relative to B, one multiplies by P_B^{-1}.

More generally, if we have bases B and C, then there is a matrix that allows one to convert between the two coordinate systems:

*If B and C are bases of $\mathbb{R}^n$, then the **change-of-coordinates** matrix, denoted $\underset{C\leftarrow B}{P}$, allows one to convert between the two coordinate systems. That is*

$$[\boldsymbol{x}]_C = \underset{C\leftarrow B}{P}\,[\boldsymbol{x}]_B$$

The columns of $\underset{C\leftarrow B}{P}$ express each vector of B in terms of C.

A cute fact is that $\underset{C\leftarrow B}{P} = P_C^{-1}P_B$. This is not surprising: it says that to convert from B to C, one can convert from B to standard coordinates, and then from there to C.

EXAMPLE. *If $B = \{(1,0),(1,3)\}$ and $C = \{(1,-2),(3,-5)\}$, then* $\underset{C\leftarrow B}{P} = \begin{bmatrix} -5 & -14 \\ 2 & 5 \end{bmatrix}$

Practice

6.1. Consider the matrix

$$F = \begin{bmatrix} 2 & -1 & 0 & -3 \\ 12 & -6 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give the dimension and a basis of the column space of F.

(b) Give the dimension and a basis of the null space of F.

(c) Give the dimension and a basis of the row space of F.

6.2. Assume A and B are 2×2 matrices. For each of the following, either prove or disprove. (That is, either explain why we know it's always true, or give an example that shows it is not always true.)

(a) If both A and B have rank 2, then so does the product AB.

(b) If both A and B have rank 1, then so does the product AB.

(c) If both A and B have rank 0, then so does the product AB.

6.3. Consider basis $B = \{(1,3), (-4,1)\}$ of $\mathbb{R}^2$.

(a) If $[\boldsymbol{x}]_B = (7, -2)$, what is $\boldsymbol{x}$?

(b) If $\boldsymbol{y} = (-5, 5)$, what is $[\boldsymbol{y}]_B$?

(c) If $\boldsymbol{z} = [\boldsymbol{z}]_B$, what is $\boldsymbol{z}$?

Solutions to Practice Exercieses

6.1. (a) Dimension 2. Example basis: the first and third columns of F.

(b) Dimension 2. Example basis: $\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3/2 \\ 0 \\ -25 \\ 1 \end{bmatrix}$

(c) Dimension 2. Example basis: the first two rows of F.

6.2. (a) True. For 2×2 matrices, having rank 2 is equivalent to being invertible, and we know from Chapter 3 that the product of invertible matrices is invertible.

(b) Not necessarily true. Consider $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. They have rank 1 but their product has rank 0.

(c) True. The only matrix that has rank 0 is the all-zero matrix.

6.3. (a) $(15, 19)$

(b) $(15/13, 20/13)$

(c) $(0, 0)$