7 Vector Spaces and Linear Transforms

In this chapter we show that the idea of a vector and of a matrix transform can be generalized.

7.1 Vector Spaces

A ***vector space*** *is a collection of vector-like-objects with operations addition and scalar multiplication defined that obey the "usual" vector laws.*

The full list of laws, also known as the **axioms** of a vector space, is given on the next page. In short, they say that addition and scalar multiplication behave like they do for ordinary vectors.

More specifically, the axioms start with the requirement that the vector space is closed under the two operations: that is, if you take two objects in the space and add them, the sum is in the space; if you take an object in the space and scale it by some real number, the scaled object is in the space. Other **axioms** include that:

- addition is **commutative**, **associative**, and has **negation**;
- the **0** vector and 1 scalar behave as **identities**; and
- addition and scalar multiplication **distribute**.

EXAMPLE. $\mathbb{R}^n$, *with addition and scalar multiplication as we've been doing, is a vector space.*

To provide some examples, let us define:

- $\mathbb{P}_n$ *is the set of all polynomials in* t *of degree at most* n
- $\mathbb{P}$ *is the set of all polynomials in* t
- $C[t]$ *is the set of all continuous functions in variable* t *with domain all of* $\mathbb{R}$
- M_n *is the set of all* $n \times n$ *matrices*

EXAMPLE. *To see that $\mathbb{P}_n$ is a vector space, note, for example, that if one adds two polynomials the result is a polynomial, and its degree cannot be larger than both summands, and thus $\mathbb{P}_n$ is closed under addition. The zero of the vector space is the* 0 *polynomial. Similarly $\mathbb{P}$, the set of all polynomials, is a vector space.*
From calculus we know that if you add continuous functions then the result is a continuous function. So $C[t]$ is a vector space. (The zero constant function plays the role of the zero vector.) And we saw earlier that one can add and scale matrices; thus M_n is a vector space.

Here are the promised axioms of a vector space V. As is standard, we will just talk about "vectors" rather than "vector-like objects". Hopefully this is not too confusing.

AXIOMS *For all vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in V and all (real) scalars c and d:*

1) The sum $\boldsymbol{u} + \boldsymbol{v}$ is in V

2) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

3) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

4) There is a vector $\mathbf{0}$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$

5) There is a vector $-\boldsymbol{u}$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$

6) The scalar multiple $c\boldsymbol{u}$ is in V

7) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$

8) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$

9) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$

10) $1\boldsymbol{u} = \boldsymbol{u}$

It should be noted that what we have defined as a vector space is sometimes called a **real vector space**, because the scalars are restricted to being real numbers. A **complex vector space** would be one where the scalars are allowed to be complex numbers.

7.2 Spans, Linear Independence, and Subspaces

Most of the concepts that we introduced for $\mathbb{R}^n$ carry over to an abstract vector space. In particular, one can talk about the span of a set of vector-like-objects as before and a subspace. A consequence of the definition of a subspace is that a subspace is a vector space in its own right, using the same operations.

EXAMPLE. *Recall that* $\mathbb{P}_n$ *is the set of all polynomials of degree at most* n. *This is a subspace of the space* $\mathbb{P}$ *of all polynomials. And* $\mathbb{P}$ *is a subspace of the space* $C[t]$ *of continuous functions. Another subspace of subspace of* $C[t]$ *is the set of continuous functions such that* $\int_{-\infty}^{\infty} f(t)\,dt = 0$.

We can also define linearly independent, basis, and dimension as before. So for example, a **basis** B of a vector space is a set such that (i) B is linearly independent, and (ii) the span of B is the whole vector space. The dimension of the vector space is the size of a basis.

EXAMPLE. $\mathbb{P}_n$ *has dimension* $n+1$. *A basis is the set* $\{1, t, t^2, \ldots, t^n\}$.

EXAMPLE. *The set* $C[t]$ *of continuous functions has infinite dimension.*

EXAMPLE. *The set* M_2 *of* 2×2 *matrices has dimension* 4. *A basis is, for example, the quartet of* 2×2 *matrices that have three* 0*'s and one* 1. *(Equivalently, to specify a* 2×2 *matrix one has to give* 4 *numbers.)*

EXAMPLE. *In calculus one might consider the equation* $f'' = -f$; *that is, all functions whose second derivative is the negative of the function. One solution to this equation is* $\sin t$; *another is* $\cos t$. *Indeed, any linear combination of* $\sin t$ *and* $\cos t$ *is a solution. It can be shown that this gives all the solutions. That is, the set of all functions* $f(t)$ *that obey the differential equation has basis* $\{\sin t, \cos t\}$.

7.3 Linear Transforms

We saw matrix transforms earlier. If A is an $m \times n$ matrix, then the matrix transform $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. This can be generalized:

> *A* ***linear transform*** T *is a function from one vector space to another vector space. It is required to obey two rules. For all vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in the domain vector space and all reals* c*:*
>
> *(1)* $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$, *and*
>
> *(2)* $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$.

That is, a linear transform has the property: if you add first and then transform, you get the same result as if you transform first and then add; if you scale first and then transform, you get the same result as if you transform first and then scale. One can easily check that matrix multiplication obeys these two rules; that is:

Fact 7.1 *Every matrix transform is a linear transform.*

The null space of a linear transform is the set of all vectors that are mapped to $\mathbf{0}$*; it is often called the* ***kernel*** *of the transform.*

EXAMPLE. *For example, differentiation is a linear transform from the polynomial space* $\mathbb{P}_n$ *to the polynomial space* $\mathbb{P}_{n-1}$*. Its kernel is the set of all constants.*

It can be shown that:

Fact 7.2 *For any linear transform from vector space* V *to subspace* W*:*
1) The kernel is a subspace of V*.*
2) The range is a subspace of W*.*
3) The dimension of the kernel and the dimension of the range sum to the dimension of V*.*

Practice

7.1. In each of the following, state whether it is a vector space. Justify your answer. For the cases where the set is a space, give the dimension and a basis.

(a) the set of all polynomials with degree exactly 1

(b) the set of all 2×2 matrices with determinant 2

(c) the set of all diagonal 3×3 matrices

(d) the set of all vectors in $\mathbb{R}^4$ whose entries sum to 0

(e) the set of all antiderivatives of $f(x) = x^5$

7.2. Show that the axioms of a vector space imply that: for any vector $\boldsymbol{u}$ its additive inverse is unique.

7.3. Show that for any linear transform T it holds that $T\mathbf{0} = \mathbf{0}$.

Solutions to Practice Exercieses

7.1. (a) No; does not include zero

(b) No; does not include zero

(c) Vector space. Dimension 3. The trio of matrices with 1 in one diagonal position and 0's elsewhere.

(d) Vector space. Dimension 3. For example, $(1, -1, 0, 0)$, $(1, 0, -1, 0)$, $(1, 0, 0, -1)$.

(e) No; does not include zero

7.2. The standard way to show that something is unique is to suppose there are two different possibilities and prove them equal. So, suppose that $\boldsymbol{u}$ has additive inverses $\boldsymbol{a}$ and $\boldsymbol{b}$. This means that $\boldsymbol{u} + \boldsymbol{a} = \mathbf{0}$ and $\boldsymbol{u} + \boldsymbol{b} = \mathbf{0}$. Consider $X = (\boldsymbol{a} + \boldsymbol{u}) + \boldsymbol{b}$. We know $X = (\boldsymbol{u} + \boldsymbol{a}) + \boldsymbol{b}$ by Axiom 2, and so $X = \mathbf{0} + \boldsymbol{b} = \boldsymbol{b}$ by Axiom 4. On the other hand, $X = \boldsymbol{a} + (\boldsymbol{u} + \boldsymbol{b})$ by Axiom 3. So $X = \boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$. Thus we have shown that $\boldsymbol{a} = \boldsymbol{b}$. That is, the inverse is unique.

7.3. One way to see this is that $T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0})$ by simplification, but $T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$ by the rules of a linear transform. So $T(\mathbf{0}) + T(\mathbf{0}) = T(\mathbf{0})$, which means that $T(\mathbf{0}) = \mathbf{0}$.