8 Eigenvalues

In this chapter we consider the eigenvalues and eigenvectors of a matrix.

8.1 Eigenvalues and Eigenvectors

*An **eigenvector** of a square matrix A is a nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some scalar λ called an **eigenvalue**.*

We will soon discuss how to find them and indeed whether they always exist. But first an example:

EXAMPLE. *Consider matrix* $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. *Then* $\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *is an eigenvector for* $\lambda_1 = -2$, *since* $A\boldsymbol{x}_1 = (2, -2)$; *and* $\boldsymbol{x}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ *is an eigenvector for* $\lambda_2 = 3$, *since* $A\boldsymbol{x}_2 = (-9, 6)$.

Now, in general the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is equivalent to $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$. That is, the eigenvector $\boldsymbol{x}$ is in the null space of $A - \lambda I$. So it matters when the null space of $A - \lambda I$ is nontrivial, or equivalently:

Fact 8.1 *The value λ is an eigenvalue if and only if the matrix $A - \lambda I$ is not invertible.*

In particular, it follows that 0 is an eigenvalue of A if and only if A is not invertible.

*The **characteristic polynomial** of a matrix A is defined as the determinant of the matrix $A - \lambda I$.*

It can be shown that if A is $n \times n$ then the characteristic polynomial is a polynomial of degree n in variable λ. Further, the above discussion shows that:

Fact 8.2 *The eigenvalues are the roots of the characteristic polynomial.*

EXAMPLE. *Consider again the matrix* $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. *This matrix has characteristic polynomial* $(13-\lambda)(-12-\lambda) - (15)(-10) = \lambda^2 - \lambda - 6$. *This polynomial has roots* $\lambda = 3$ *and* $\lambda = -2$.

The characteristic polynomial approach gives us a method to find the eigenvalues, though for large matrices potentially the method is inefficient and/or requires numerical methods to find the roots.

> *The **eigenspace** of* λ *is defined as all eigenvectors corresponding to* λ *along with the zero vector.*

Equivalently, the eigenspace of eigenvalue λ is the null space of $A - \lambda I$.

EXAMPLE. *Consider again* $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. *The eigenvalues are* $\lambda_1 = 3$ *and* $\lambda_2 = -2$. *For* λ_1, *we need a vector in the null space of* $\begin{bmatrix} 15 & 15 \\ -10 & -10 \end{bmatrix}$; *for example* $(-1, 1)$. *For* λ_2, *we need a vector in the null space of* $\begin{bmatrix} 10 & 15 \\ -10 & -15 \end{bmatrix}$; *for example* $(-3, 2)$.

> **Fact 8.3** *The eigenvalues of a diagonal matrix are its diagonal entries. The vectors* $\boldsymbol{e}_i$ *are its eigenvectors.*

We used ad hoc techniques in the above examples, but in general the problem then is to find the null space of $A - \lambda I$. And we know from previous chapters how to find null spaces!

EXAMPLE. *Consider the matrix*

$$\begin{bmatrix} 5 & -4 & -4 \\ -8 & 9 & 8 \\ 10 & -10 & -9 \end{bmatrix}$$

It can be calculated that this matrix has characteristic polynomial $-\lambda^3 + 5\lambda^2 - 7\lambda + 3$. *The roots of the characteristic polynomial are (according to software)* $3, 1, 1$. *Finally, it can be calculated that the eigenspaces have bases as follows:*

$$\lambda = 1\text{:} \quad \begin{bmatrix} 4 & -4 & -4 \\ -8 & 8 & 8 \\ 10 & -10 & -10 \end{bmatrix} \quad \text{has null space with basis} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 3 : \quad \begin{bmatrix} 2 & -4 & -4 \\ -8 & 6 & 8 \\ 10 & -10 & -12 \end{bmatrix} \textit{ has null space with basis } \quad \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix}$$

8.2 Properties of Eigenvalues and Eigenvectors

One useful property of eigenvectors is the following observation. If one has a vector $\boldsymbol{v}$ expressed as a linear combination of eigenvectors of matrix A, then multiplication by A is quick. For, say vector $\boldsymbol{v} = \sum_i a_i \boldsymbol{x}_i$, where $A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$; then

$$A\boldsymbol{v} = \sum_i a_i \lambda_i \boldsymbol{x}_i$$

This property is especially useful if there is a basis for the space consisting of eigenvectors.

*An $n \times n$ matrix is defined to have a **full set of eigenvectors** if it has n linearly independent eigenvectors.*

It turns out that if all n eigenvalues are different, then such a full set is guaranteed. This follows from the more general fact that we do not prove:

Fact 8.4 *Eigenvectors for distinct eigenvalues are linearly independent.*

Applying a matrix transform repeatedly is equivalent to transforming with the power of the matrix. There is a simple relationship between the eigenvalues of matrix A and matrix A^k:

Fact 8.5 *If matrix A has eigenvalues λ_i, then the power A^k has eigenvalues λ_i^k. Moreover, the eigenvectors are the same.*

Proof. *Assume $A\boldsymbol{x} = \lambda_i \boldsymbol{x}$. Then $A^k \boldsymbol{x} = A^{k-1}\lambda_i \boldsymbol{x} = \cdots = \lambda_i^k \boldsymbol{x}$. So $\boldsymbol{x}$ is also an eigenvector of A^k, but with eigenvalue λ_i^k.*

We finish this section with some (remarkable!) facts that can be deduced from properties of the sum or product of the roots of a polynomial.

*The **trace** of a matrix is defined as the sum of the diagonal entries.*

Fact 8.6 *For any matrix A,*
(a) the determinant of A equals the product of its eigenvalues.
(b) the trace of A equals the sum of its eigenvalues.

8.3 Similarity and Diagonalization

*Matrices A and B are defined to be **similar** if there is an invertible matrix P such that multiplying A on the one side by P and on the other side by P^{-1} gives B. Note that the definition is symmetric: if $B = P^{-1}AP$ then $A = PBP^{-1}$.*

Using the fact that the product of determinants equals the determinant of the product, it can be shown that:

Fact 8.7 *If matrices A and B are similar, then they have the same characteristic polynomial.*

However, while similar matrices have the same eigenvalues, they need not (and usually do not) have the same eigenvectors.

*A matrix A is defined to be **diagonalizable** if it is similar to a diagonal matrix.*

By the above fact, if A is similar to diagonal matrix D, then the diagonal entries of D must be the eigenvalues of A. The following result shows how to diagonalize a matrix if it has a full set of eigenvectors:

Fact 8.8 *If matrix A has a full set of eigenvectors, then $A = PDP^{-1}$, where matrix P has the eigenvectors of A as its columns, and diagonal matrix D has the eigenvalues of A on its diagonal (in the same order).*

PROOF. *Say $n \times n$ matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) with eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ that are linearly independent. Then, because each column of P is an eigenvector of A, we have that the columns of the product AP can be written as:*

$$AP = [\lambda_1 \boldsymbol{v}_1 \ \lambda_2 \boldsymbol{v}_2 \ \cdots \ \lambda_k \boldsymbol{v}_k]$$

If one multiplies any matrix on the right by a diagonal matrix, it multiplies the columns *by those diagonal entries. That is,*

$$PD = [\lambda_1 \boldsymbol{v}_1 \ \lambda_2 \boldsymbol{v}_2 \ \cdots \ \lambda_k \boldsymbol{v}_k]$$

That is, we have shown that

$$AP = PD.$$

Since we were told the eigenvectors are linearly independent, the matrix P is invertible. Thus $(AP)P^{-1} = (PD)P^{-1}$, and so $A = PDP^{-1}$, as required.

Example. *Consider* $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. *Then* $P = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$ *and* $D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$. *The reader should check the product PDP^{-1}.*

Diagonalization is very useful in applications. One little application is

> **Fact 8.9** *If $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$.*

The proof is left as an exercise.

8.4 Complex Eigenvalues

Recall that i denotes the square-root of -1. Complex numbers have the form $a + bi$, where a and b are real numbers. If $\lambda = a+bi$, then its **(complex) conjugate** is defined to be $\bar{\lambda} = a - bi$.

We do not prove the following result; but, for example, in a 2×2 matrix, its truth follows from the quadratic formula.

> **Fact 8.10** *If λ is a complex eigenvalue of A, then so is its conjugate $\bar{\lambda}$.*

Example. *Consider the matrix* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. *The characteristic polynomial is given by $(a-\lambda)^2 + b^2$; thus the matrix has eigenvalues $\lambda = a \pm bi$.*
As a matrix transform, the above matrix represents scaling by $|\lambda| = \sqrt{a^2+b^2}$ and rotation through the angle $\arctan b/a$.

An important fact is the following. We omit the proof:

Fact 8.11 *A real symmetric matrix has only real eigenvalues.*

Practice

8.1. For the following matrices, find all eigenvalues and a basis for each eigenspace.

$$J = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & 3 & 3 \end{bmatrix}$$

8.2. Prove that: if matrix A is similar to matrix B, and matrix B is similar to matrix C, then A is similar to C.

8.3. Diagonalize the matrix J from the first question by computing P, D, and P^{-1}.

Solutions to Practice Exercieses

8.1. J has eigenvalue 2 with eigenvector $(3, 1)$ and eigenvalue 4 with eigenvector $(1, 1)$.
K has eigenvalue 3 with eigenvector $(1, 1, 1)$ and eigenvalue 0 with basis $(1, -1, 0)$, $(1, 0, -1)$.
L has eigenvalue 2 with eigenvector $(0, -\frac{1}{3}, 1)$ and eigenvalue 6 with basis $(0, 1, 1)$.

8.2. Assume $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Thus $A = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP)$.

8.3. One solution is given by

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

Another solution is obtained by interchanging the columns of P, interchanging the entries of D, and interchanging the rows of P^{-1}.