8 Eigenvalues

In this chapter we consider the eigenvalues and eigenvectors of a matrix.

8.1 Eigenvalues and Eigenvectors

An eigenvector of a square matrix $A$ is a nonzero vector $x$ such that $Ax = \lambda x$ for some scalar $\lambda$ called an eigenvalue.

We will soon discuss how to find them and indeed whether they always exist. But first an example:

**Example.** Consider matrix $A = \begin{bmatrix} 1 & 3 \\ -10 & -12 \end{bmatrix}$. Then $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = -2$, since $Ax_1 = (2, -2)$; and $x_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 3$, since $Ax_2 = (-9, 6)$.

Now, in general the equation $Ax = \lambda x$ is equivalent to $(A - \lambda I)x = 0$. That is, the eigenvector $x$ is in the null space of $A - \lambda I$. So it matters when the null space of $A - \lambda I$ is nontrivial, or equivalently:

**Fact 8.1** The value $\lambda$ is an eigenvalue if and only if the matrix $A - \lambda I$ is not invertible.

In particular, it follows that 0 is an eigenvalue of $A$ if and only if $A$ is not invertible.

The characteristic polynomial of a matrix $A$ is defined as the determinant of the matrix $A - \lambda I$.

It can be shown that if $A$ is $n \times n$ then the characteristic polynomial is a polynomial of degree $n$ in variable $\lambda$. Further, the above discussion shows that:

**Fact 8.2** The eigenvalues are the roots of the characteristic polynomial.
Example. Consider again the matrix \( A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix} \). This matrix has characteristic polynomial \((13 - \lambda)(-12 - \lambda) - (15)(-10) = \lambda^2 - \lambda - 6\). This polynomial has roots \( \lambda = 3 \) and \( \lambda = -2 \).

The characteristic polynomial approach gives us a method to find the eigenvalues, though for large matrices potentially the method is inefficient and/or requires numerical methods to find the roots.

The eigenspace of \( \lambda \) is defined as all eigenvectors corresponding to \( \lambda \) along with the zero vector.

Equivalently, the eigenspace of eigenvalue \( \lambda \) is the null space of \( A - \lambda I \).

Example. Consider again \( A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix} \). The eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = -2 \).

For \( \lambda_1 \), we need a vector in the null space of \( \begin{bmatrix} 15 & 15 \\ -10 & -10 \end{bmatrix} \); for example \((-1, 1)\). For \( \lambda_2 \), we need a vector in the null space of \( \begin{bmatrix} 10 & 15 \\ -10 & -15 \end{bmatrix} \); for example \((-3, 2)\).

Fact 8.3 The eigenvalues of a diagonal matrix are its diagonal entries. The vectors \( e_i \) are its eigenvectors.

We used ad hoc techniques in the above examples, but in general the problem then is to find the null space of \( A - \lambda I \). And we know from previous chapters how to find null spaces!

Example. Consider the matrix

\[
\begin{bmatrix} 5 & -4 & -4 \\ -8 & 9 & 8 \\ 10 & -10 & -9 \end{bmatrix}
\]

It can be calculated that this matrix has characteristic polynomial \(-\lambda^3 + 5\lambda^2 - 7\lambda + 3\). The roots of the characteristic polynomial are (according to software) 3, 1, 1. Finally, it can be calculated that the eigenspaces have bases as follows:

\[\lambda = 1: \begin{bmatrix} 4 & -4 & -4 \\ -8 & 8 & 8 \\ 10 & -10 & -10 \end{bmatrix} \text{ has null space with basis } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\]
8.2 Properties of Eigenvalues and Eigenvectors

One useful property of eigenvectors is the following observation. If one has a vector $v$ expressed as a linear combination of eigenvectors of matrix $A$, then multiplication by $A$ is quick. For, say vector $v = \sum a_i x_i$, where $A x_i = \lambda_i x_i$; then

$$Av = \sum a_i \lambda_i x_i$$

This property is especially useful if there is a basis for the space consisting of eigenvectors.

An $n \times n$ matrix is defined to have a **full set of eigenvectors** if it has $n$ linearly independent eigenvectors.

It turns out that if all $n$ eigenvalues are different, then such a full set is guaranteed. This follows from the more general fact that we do not prove:

**Fact 8.4** Eigenvectors for distinct eigenvalues are linearly independent.

Applying a matrix transform repeatedly is equivalent to transforming with the power of the matrix. There is a simple relationship between the eigenvalues of matrix $A$ and matrix $A^k$:

**Fact 8.5** If matrix $A$ has eigenvalues $\lambda_i$, then the power $A^k$ has eigenvalues $\lambda_i^k$. Moreover, the eigenvectors are the same.

Proof. Assume $A x = \lambda_i x$. Then $A^k x = A^{k-1} \lambda_i x = \cdots = \lambda_i^k x$. So $x$ is also an eigenvector of $A^k$, but with eigenvalue $\lambda_i^k$.

We finish this section with some (remarkable!) facts that can be deduced from properties of the sum or product of the roots of a polynomial.

The **trace** of a matrix is defined as the sum of the diagonal entries.
For any matrix $A$,
(a) the determinant of $A$ equals the product of its eigenvalues.
(b) the trace of $A$ equals the sum of its eigenvalues.

### 8.3 Similarity and Diagonalization

Matrices $A$ and $B$ are defined to be **similar** if there is an invertible matrix $P$ such that multiplying $A$ on the one side by $P$ and on the other side by $P^{-1}$ gives $B$. Note that the definition is symmetric: if $B = P^{-1}AP$ then $A = PBP^{-1}$.

Using the fact that the product of determinants equals the determinant of the product, it can be shown that:

**Fact 8.7** If matrices $A$ and $B$ are similar, then they have the same characteristic polynomial.

However, while similar matrices have the same eigenvalues, they need not (and usually do not) have the same eigenvectors.

A matrix $A$ is defined to be **diagonalizable** if it is similar to a diagonal matrix.

By the above fact, if $A$ is similar to diagonal matrix $D$, then the diagonal entries of $D$ must be the eigenvalues of $A$. The following result shows how to diagonalize a matrix if it has a full set of eigenvectors:

**Fact 8.8** If matrix $A$ has a full set of eigenvectors, then $A = PDP^{-1}$, where matrix $P$ has the eigenvectors of $A$ as its columns, and diagonal matrix $D$ has the eigenvalues of $A$ on its diagonal (in the same order).

**Proof.** Say $n \times n$ matrix $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) with eigenvectors $v_1, \ldots, v_n$ that are linearly independent. Then, because each column of $P$ is an eigenvector of $A$, we have that the columns of the product $AP$ can be written as:

$$AP = [\lambda_1v_1 \lambda_2v_2 \cdots \lambda_kv_k]$$

If one multiplies any matrix on the right by a diagonal matrix, it multiplies the columns
by those diagonal entries. That is,

$$PD = [\lambda_1 v_1 \lambda_2 v_2 \cdots \lambda_k v_k]$$

That is, we have shown that

$$AP = PD.$$ 

Since we were told the eigenvectors are linearly independent, the matrix $P$ is invertible.

Thus $(AP)P^{-1} = (PD)P^{-1}$, and so $A = PDP^{-1}$, as required.

**Example.** Consider $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. Then $P = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$. The reader should check the product $PDP^{-1}$.

Diagonalization is very useful in applications. One little application is

**Fact 8.9** If $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$.

The proof is left as an exercise.

### 8.4 Complex Eigenvalues

Recall that $i$ denotes the square-root of $-1$. Complex numbers have the form $a + bi$, where $a$ and $b$ are real numbers. If $\lambda = a + bi$, then its (complex) conjugate is defined to be $\bar{\lambda} = a - bi$.

We do not prove the following result; but, for example, in a $2 \times 2$ matrix, its truth follows from the quadratic formula.

**Fact 8.10** If $\lambda$ is a complex eigenvalue of $A$, then so is its conjugate $\bar{\lambda}$.

**Example.** Consider the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. The characteristic polynomial is given by $(a - \lambda)^2 + b^2$; thus the matrix has eigenvalues $\lambda = a \pm bi$.

As a matrix transform, the above matrix represents scaling by $|\lambda| = \sqrt{a^2 + b^2}$ and rotation through the angle $\arctan b/a$.

An important fact is the following. We omit the proof:
Fact 8.11 A real symmetric matrix has only real eigenvalues.

Practice

8.1. For the following matrices, find all eigenvalues and a basis for each eigenspace.

\[ J = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \quad K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & 3 & 3 \end{bmatrix} \]

8.2. Prove that: if matrix \( A \) is similar to matrix \( B \), and matrix \( B \) is similar to matrix \( C \), then \( A \) is similar to \( C \).

8.3. Diagonalize the matrix \( J \) from the first question by computing \( P, D, \) and \( P^{-1} \).

Solutions to Practice Exercises

8.1. \( J \) has eigenvalue 2 with eigenvector \((3, 1)\) and eigenvalue 4 with eigenvector \((1, 1)\).

\( K \) has eigenvalue 3 with eigenvector \((1, 1, 1)\) and eigenvalue 0 with basis \((1, -1, 0), (1, 0, -1)\).

\( L \) has eigenvalue 2 with eigenvector \((0, -\frac{1}{3}, 1)\) and eigenvalue 6 with basis \((0, 1, 1)\).

8.2. Assume \( A = P^{-1}BP \) and \( B = Q^{-1}CQ \). Thus \( A = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP) \).

8.3. One solution is given by

\[ P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \]

Another solution is obtained by interchanging the columns of \( P \), interchanging the entries of \( D \), and interchanging the rows of \( P^{-1} \).