8 Eigenvalues

In this chapter we consider the eigenvalues and eigenvectors of a matrix.

8.1 Eigenvalues and Eigenvectors

An eigenvector of a square matrix $A$ is a nonzero vector $x$ such that $Ax = \lambda x$ for some scalar $\lambda$ called an eigenvalue.

We will soon discuss how to find them and indeed whether they always exist. But first an example:

- Example. Consider matrix $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. Then $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = -2$, since $Ax_1 = (2, -2)$; and $x_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 3$, since $Ax_2 = (-9, 6)$.

Now, in general the equation $Ax = \lambda x$ is equivalent to $(A - \lambda I)x = 0$. That is, the eigenvector $x$ is in the null space of $A - \lambda I$. So it matters when the null space of $A - \lambda I$ is nontrivial, or equivalently:

**Fact 8.1** The value $\lambda$ is an eigenvalue if and only if the matrix $A - \lambda I$ is not invertible.

In particular, it follows that 0 is an eigenvalue of $A$ if and only if $A$ is not invertible.

The characteristic polynomial of a matrix $A$ is defined as the determinant of the matrix $A - \lambda I$.

It can be shown that if $A$ is $n \times n$ then the characteristic polynomial is a polynomial of degree $n$ in variable $\lambda$. Further, the above discussion shows that:

**Fact 8.2** The eigenvalues are the roots of the characteristic polynomial.
Example. Consider again the matrix $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. This matrix has characteristic polynomial $(13 - \lambda)(-12 - \lambda) - (15)(-10) = \lambda^2 - \lambda - 6$. This polynomial has roots $\lambda = 3$ and $\lambda = -2$.

The characteristic polynomial approach gives us a method to find the eigenvalues, though for large matrices potentially the method is inefficient and/or requires numerical methods to find the roots.

The eigenspace of $\lambda$ is defined as all eigenvectors corresponding to $\lambda$ along with the zero vector.

Equivalently, the eigenspace of eigenvalue $\lambda$ is the null space of $A - \lambda I$.

Example. Consider again $A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$.

For $\lambda_1$, we need a vector in the null space of $\begin{bmatrix} 15 & 15 \\ -10 & -10 \end{bmatrix}$; for example $(-1, 1)$. For $\lambda_2$, we need a vector in the null space of $\begin{bmatrix} 10 & 15 \\ -10 & -15 \end{bmatrix}$; for example $(-3, 2)$.

Fact 8.3 The eigenvalues of a diagonal matrix are its diagonal entries. The vectors $e_i$ are its eigenvectors.

We used ad hoc techniques in the above examples, but in general the problem then is to find the null space of $A - \lambda I$. And we know from previous chapters how to find null spaces!

Example. Consider the matrix

$\begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$

It can be calculated that this matrix has characteristic polynomial $-\lambda^3 + 5\lambda^2 - 7\lambda + 3$. The roots of the characteristic polynomial are (according to software) 3, 1, 1. Finally, it can be calculated that the eigenspaces have bases as follows:

$\lambda = 1$: $\begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix}$ has null space with basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
### 8.2 Properties of Eigenvalues and Eigenvectors

One useful property of eigenvectors is the following observation. If one has a vector $v$ expressed as a linear combination of eigenvectors of matrix $A$, then multiplication by $A$ is quick. For, say vector $v = \sum_i a_i x_i$, where $A x_i = \lambda_i x_i$; then

$$A v = \sum_i a_i \lambda_i x_i$$

This property is especially useful if there is a basis for the space consisting of eigenvectors.

An $n \times n$ matrix is defined to have a **full set of eigenvectors** if it has $n$ linearly independent eigenvectors.

It turns out that if all $n$ eigenvalues are different, then such a full set is guaranteed. This follows from the more general fact that we do not prove:

**Fact 8.4** Eigenvectors for distinct eigenvalues are linearly independent.

Applying a matrix transform repeatedly is equivalent to transforming with the power of the matrix. There is a simple relationship between the eigenvalues of matrix $A$ and matrix $A^k$:

**Fact 8.5** If matrix $A$ has eigenvalues $\lambda_i$, then the power $A^k$ has eigenvalues $\lambda_i^k$. Moreover, the eigenvectors are the same.

**Proof.** Assume $A x = \lambda_i x$. Then $A^k x = A^{k-1} \lambda_i x = \cdots = \lambda_i^k x$. So $x$ is also an eigenvector of $A^k$, but with eigenvalue $\lambda_i^k$.

We finish this section with some (remarkable!) facts that can be deduced from properties of the sum or product of the roots of a polynomial.

**The trace of a matrix is defined as the sum of the diagonal entries.**
8.3 Similarity and Diagonalization

Matrices $A$ and $B$ are defined to be similar if there is an invertible matrix $P$ such that multiplying $A$ on the one side by $P$ and on the other side by $P^{-1}$ gives $B$. Note that the definition is symmetric: if $B = P^{-1}AP$ then $A = PB^{-1}$.

Using the fact that the product of determinants equals the determinant of the product, it can be shown that:

Fact 8.7 If matrices $A$ and $B$ are similar, then they have the same characteristic polynomial.

However, while similar matrices have the same eigenvalues, they need not (and usually do not) have the same eigenvectors.

A matrix $A$ is defined to be diagonalizable if it is similar to a diagonal matrix.

By the above fact, if $A$ is similar to diagonal matrix $D$, then the diagonal entries of $D$ must be the eigenvalues of $A$. The following result shows how to diagonalize a matrix if it has a full set of eigenvectors:

Fact 8.8 If matrix $A$ has a full set of eigenvectors, then $A = PDP^{-1}$, where matrix $P$ has the eigenvectors of $A$ as its columns, and diagonal matrix $D$ has the eigenvalues of $A$ on its diagonal (in the same order).

Proof. Say $n \times n$ matrix $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) with eigenvectors $v_1, \ldots, v_n$ that are linearly independent. Then, because each column of $P$ is an eigenvector of $A$, we have that the columns of the product $AP$ can be written as:

$$AP = [\lambda_1 v_1 \lambda_2 v_2 \cdots \lambda_k v_k]$$

If one multiplies any matrix on the right by a diagonal matrix, it multiplies the columns
by those diagonal entries. That is,

\[ PD = [\lambda_1 v_1 \ \lambda_2 v_2 \ \cdots \ \lambda_k v_k] \]

That is, we have shown that

\[ AP = PD. \]

Since we were told the eigenvectors are linearly independent, the matrix \( P \) is invertible. Thus \( (AP)P^{-1} = (PD)P^{-1} \), and so \( A = PDP^{-1} \), as required.

Example. Consider \( A = \begin{bmatrix} 13 & 15 \\ -10 & -12 \end{bmatrix} \). Then \( P = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix} \) and \( D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \). The reader should check the product \( PDP^{-1} \).

Diagonalization is very useful in applications. One little application is

**Fact 8.9** If \( A = PDP^{-1} \) then \( A^k = PD^kP^{-1} \).

The proof is left as an exercise.

### 8.4 Complex Eigenvalues

Recall that \( i \) denotes the square-root of \(-1\). Complex numbers have the form \( a + bi \), where \( a \) and \( b \) are real numbers. If \( \lambda = a + bi \), then its (complex) conjugate is defined to be \( \bar{\lambda} = a - bi \).

We do not prove the following result; but, for example, in a \( 2 \times 2 \) matrix, its truth follows from the quadratic formula.

**Fact 8.10** If \( \lambda \) is a complex eigenvalue of \( A \), then so is its conjugate \( \bar{\lambda} \).

Example. Consider the matrix \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \). The characteristic polynomial is given by \( (a - \lambda)^2 + b^2 \); thus the matrix has eigenvalues \( \lambda = a \pm bi \).

As a matrix transform, the above matrix represents scaling by \( |\lambda| = \sqrt{a^2 + b^2} \) and rotation through the angle \( \arctan b/a \).

An important fact is the following. We omit the proof:
Fact 8.11 A real symmetric matrix has only real eigenvalues.

Practice

8.1. For the following matrices, find all eigenvalues and a basis for each eigenspace.

\[ J = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \quad K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & 3 & 3 \end{bmatrix} \]

8.2. Prove that: if matrix $A$ is similar to matrix $B$, and matrix $B$ is similar to matrix $C$, then $A$ is similar to $C$.

8.3. Diagonalize the matrix $J$ from the first question by computing $P$, $D$, and $P^{-1}$.

Solutions to Practice Exercises

8.1. $J$ has eigenvalue 2 with eigenvector $(3, 1)$ and eigenvalue 4 with eigenvector $(1, 1)$.

$K$ has eigenvalue 3 with eigenvector $(1, 1, 1)$ and eigenvalue 0 with basis $(1, -1, 0)$, $(1, 0, -1)$.

$L$ has eigenvalue 2 with eigenvector $(0, -\frac{1}{3}, 1)$ and eigenvalue 6 with basis $(0, 1, 1)$.

8.2. Assume $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Thus $A = P^{-1}Q^{-1}CQP = (QP)^{-1}CQP$.

8.3. One solution is given by

\[ P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \]

Another solution is obtained by interchanging the columns of $P$, interchanging the entries of $D$, and interchanging the rows of $P^{-1}$. 